# ESSAYS IN OPERATIONS MANAGEMENT: APPLICATIONS IN HEALTH CARE AND THE OPERATIONS-FINANCE INTERFACE

A Dissertation

by

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ABSTRACT

I present three essays pertaining to the management of supply chain risks in this dissertation. The first essay and the second essay analyze supply chain risks from a financial perspective, while the third essay analyzes supply chain risk with the objective of maximizing societal benefits in health care.

In my first essay, I consider a firm facing inventory decisions under the influence of the financial market. With stochastic analytical methods, the purpose of this essay is to examine the optimal inventory decisions under a variety of conditions. I have identified the relevant factors impacting such decisions and the firm's value. Moreover, I have studied the benefits brought by efforts to improve the random capacity of the firm. I conclude that the financial market can significantly impact both a firm's inventory decisions and process improvement incentives.

In my second essay, I model a stylized supply chain managed by a base-stock inventory policy where the decision maker holds concerns about the down-side risk of the supply chain cost. With stochastic analytical methods, the purpose of this essay is to obtain solutions of the problem of minimizing Conditional Value-at-Risk under various supply chain scenarios. I find that various supply chain parameters may influence the optimal solution and the optimality of a stock-less operation. I conclude that operating characteristics of a supply chain can shape its inventory policy when down-side risks are taken into account.

For my third essay, the purpose of this essay is to investigate the operational decisions of a medical center specializing in bone marrow transplants. Using the queuing system method, I formulate the medical center as a queuing system with random patient arrivals and departures. I find optimal decisions and efficient frontiers

regarding waiting room size and the number of transplant rooms with the objective of maximizing patient health benefits. I conclude that the design of a health care delivery system is crucial for health care institutions to sustain and improve their social impacts.

In each of the three essays, I use analytical and numerical approaches to optimize managers' decisions with respect to various sources of risk.

DEDICATION

When I told my NTU professors about going to Texas, they thought I would sleep in a ranch and get woken up by cows every morning. Taking the turboprop airplane SaaB 340 from IAH to CLL in August 2011, a vast amount of yellow dry land occupied my eyes, with zero grass in sight. I was a bit worried initially. However, things turned out to be better than I expected with new friends made and doable coursework, and I finally learned how to drive in Aggieland. Whoop!

That being said, I have experienced few challenges in my life until I entered A&M. I want to excel at what I do and find out my interests, whatever it takes. Thankfully, I met faculty members who care about me and are willing to help me, and I met peers with whom I can expect to support each other in the long term. I would like to thank Dr. Antonio Arreola-Risa, Dr. Rogelio Oliva, Dr. Gregory Heim, Dr. Chelliah Sriskandarajah, Dr. Martin Wortman, Dr. Subodha Kumar, and many others for their help during my doctoral study. I would not have gotten this far without your support. Dr. Tony, I am very grateful for your generous support, and I promise you I will publish these essays in the future. You can kick my ass if I don't.

No matter how people come and go in my life and wherever I move, I always have my parents and their love. Mr. Beining Li and Mrs. Guizhi Ma, I cannot thank you both enough for your support, and I love you. I wish you long life and prosperity. From now on, I will have your back.

Thinking about how vast the universe is in both spacial and temporal dimensions, I am smaller than a tiny drop in the ocean. I hope this dissertation is a nano-size step forward in the human civilization, which proves that I have ever existed and had a lot of fun as a researcher.

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# 1. INTRODUCTION

Operations management is beginning to interface with other disciplines, creating new fields such as the operations-finance interface and health care operations (Chopra et al., 2004). However, the development of operations and supply-chain management faces challenges such as the complexity of the systems involved, the connections with other disciplines, the uncertainties associated with the decisions, and the unique characteristics of individual application fields. For comprehensive reviews of the field of the operations-finance interface, see Hatzakis et al. (2010) and Zhao and Huchzermeier (2013). For a comprehensive review of the field of health care operations, see Langabeer (2008).

Uncertainty in operations and supply chains comes from various sources. Lee and Billington (1993) identify three sources of uncertainty: demand, process, and supply. Demand uncertainty comes from volume and product mix, process uncertainty comes from yield and capacity, and supply uncertainty comes from quality of components and delivery. In this dissertation, I adopt a prescriptive approach to manage the uncertainties in operations and supply chains related to finance and health care. We consider three different subjects and I analyze each subject separately.

Operations management is about creating value (Cohen and Kleindorfer, 1993). While value creation is essential for the survival and growth of businesses, decision-makers need to be mindful of the risks involved (Tang, 2006), which may even undermine the value of a firm in the financial market by tactical short-term decisions of the firm. How will the financial market impact inventory decisions in the presence of demand risk and supply risk? Under what circumstances should a decision maker pay special attention to financial risk? These questions motivate Section 2, in which

I describe a simple firm and evaluate the financial market value impact of inventory decisions using the Capital Asset Pricing Model (CAPM) which allows monetizing the riskiness of supply chain cash flows. I examine the problem under various scenarios to see when and how the financial market impacts inventory decisions. Moreover, I investigate when and how capacity process improvement can best enhance the firm value. I find that the correlation between demand and market returns could impact both the optimal ordering decision and the benefits of capacity process improvement. The findings of Section 2 contribute to the operations-finance literature in exploring the effects of random capacity limits on firm operating decisions and firm value and demonstrating that capacity process improvement could be worthwhile under certain situations.

Besides short-term decisions (e.g., the newsvendor model), risk also manifests itself in supply chains in the long-term. How will demand risks and supply risks impact the inventory decisions of risk-averse decision-makers? What factors contribute to the discrepancy between decisions made by risk-neutral and risk-averse decision-makers? These questions motivate Section 3, in which I describe a stylized supply-chain model managed by a base-stock inventory policy consisting of a production system and an inventory location. I consider a finite-horizon model without time-discounting. I obtain solutions to the problem of minimizing Conditional Value-at-Risk for these three scenarios. I also conduct numerical studies to investigate the impact of various parameters on the optimal solution. I discover an easy-to-use approximation of the optimal base-stock level, and I find that the optimal base-stock level increases in capacity utilization, the importance of back-orders, and risk sensitivity of the decision maker. The findings of Section 3 contribute to the operations-finance literature and the supply chain risk management literature in bringing to attention how operating characteristics can impact the inventory policy of a down-side-risk minimizing supply chain.

Not-for-profit service organizations often face the need to create value for the communities they serve (Hansmann, 1980; Dees et al., 1998; Drucker, 2001), especially in the health care industry (Himmelstein et al., 1999; Porter and Teisberg, 2006). As a result, these service organizations need to manage operational risk within their service systems to create the most value for society rather than maximizing profit. How should decision makers predict the performance of a medical center specializing in bone-marrow transplants? How should the system be designed to balance patient waiting time to be treated and patient overflow due to no waiting room available? These questions motivate Section 4 of this dissertation. In Section 4, I describe a problem faced by a medical center specializing in bone marrow transplantation and formulate a queuing-based model. I then investigate the key factors impacting the optimal decisions with analytical approaches and numerical studies. I discuss the results of the analysis and provide managerial insights. I find that myopically increasing the number of waiting rooms in the presence of a shortage of treatment capacity actually hurts patients' health benefits. The findings of Section 4 contribute to the health care operations literature in demonstrating how effective design of service-delivery systems could mitigate the effect of congestion and improve the well-beings of the society in general. In Section 5, I present a summary of this dissertation.

# 2. FINANCIAL RISK AND A NEWSVENDOR WITH RANDOM CAPACITY

## 2.1 Introduction

Companies today use strategies such as supply chain management (Christopher and Ryals, 1999) to maximize shareholder value (Lazonick and O'Sullivan, 2000) or equivalently, to maximize the value of the firm's common equity on the financial market, commonly called market capitalization or firm value. However, maximizing firm value is not easy due to the uncertainties in supply chains. In addition to demand uncertainties, supply uncertainties could manifest themselves in various forms. Some of these supply uncertainties come from technical issues; for example, Apple has reportedly cut its shipment targets for the Apple Watch in half due to an issue regarding the production of display panels (Business Insider, 2016). Other supply uncertainties may derive from business issues; for example, the launch of RIM PlayBook was delayed by one month by display shortages created by Apple (PCMag, 2016). All these uncertainties motivate firms to deliberate on their inventory and process improvement decisions, especially when these decisions have a great economic impact for a business (e.g., capital-intensive goods). It is also well illustrated that firm value can be influenced by the diverse risk profiles derived from the match between supply and demand. For example, over a two-day period, the mean stock market reaction ranges from -6.79% to -6.93% due to excess inventory (Hendricks and Singhal, 2009). It is therefore crucial to identify firm-value-maximizing inventory and process improvement decisions, the focus of this paper.

### *2.1.1 Motivation*

Despite the importance of maximizing firm value, most operations and supply chain models manage inventories by maximizing one component of firm value, namely the expected profit (e.g. Khouja, 1999), but neglect another component of firm value: the financial-market valuation of risky inventory decisions. This lack of awareness of financial-market risk may lead to sub-optimal inventory and purchasing decisions that hurt the firm's value on the financial market due to demand randomness (Singhal, 2005) and supply randomness (Hendricks and Singhal, 2008). The potential to improve firm value through better management of financial-market risk motivates us to investigate a research question: how would managers maximize firm value through inventory decisions in the presence of both demand and capacity randomness?

To study this research question, we consider a (buyer) firm which sells an item with random demand, and the item is purchased from a supplier with random capacity. Managers of the buyer firm face a newsvendor-type decision and aim to maximize the firm value. To investigate how random capacity affects inventory decisions, we consider a setting where both demand and capacity are correlated with market return (defined as the return of the portfolio that consists of all assets accessible to investors with weights proportional to market value). Positive correlation between demand and market return exists in many durable-goods industries such as the automobile industry, while negative correlation between demand and market return can be found in many low-end industries where demand improves when the economy suffers, as in the case of basic apparel with low-income target customers. On the other hand, negative correlation between capacity and market return exists in many industries where suppliers may allocate less capacity to a low-priority buyer because other buyers order more during an economic boom, while positive correlation between supplier capacity and market return can be found in the aforementioned low-end industries. Moreover, the supplier may not be in the same industry or the same country as the buyer firm, which may also lead to positive correlation between capacity and market return.

### *2.1.2 Contributions*

Our paper contributes to the literature at the operations-finance interface by underscoring the importance of capacity randomness in firm value, which agrees with and complements empirical studies in supply chain risk management (e.g., Hendricks and Singhal, 2014). We find that the firm-value-maximizing inventory ordering decision depends on capacity when the random capacity is correlated with market return. This finding is different from previous studies (Anvari, 1987; Kim and Chung, 1989) that assume infinite capacity, since we incorporate limited, random capacity and highlight its role in determining firm value.

We illustrate how the optimal inventory ordering decision responds to changes in various factors and discuss the implications of such changes. Of particular interest, we find that although higher mean capacity enables more robust supply of items, it may lead to a lower order quantity. This finding is different from Ciarallo et al. (1994), where the expected-profit-maximizing inventory ordering decision is independent of capacity, since we take into account how the financial market would price demand and capacity risks.

Moreover, we show that supplier capacity may neutralize or amplify the effect of demand-introduced systematic risk depending on the setting, which has implications in matching supply and demand to maximize firm value (Hendricks and Singhal, 2009). We also show that although both capacity and demand randomness may introduce systematic risk into a firm, the impact of capacity randomness is relatively small in size compared to that of demand randomness. However, the impact of capacity randomness may be large when the capacity utilization is high. Managers with the objective of maximizing firm value should evaluate systematic risks from demand and supply before making inventory decisions.

### *2.1.3 Literature review*

In addition to the newsvendor model (Anvari, 1987; Kim and Chung, 1989), the the Capital Asset Pricing Model (CAPM) (Sharpe, 1964; Lintner, 1965) approach has been used in multi-period supply chain settings. In continuous-review inventory models, Singhal and Raturi (1990) have shown that a firm's opportunity cost of capital depends on inventory parameters and policies; Singhal et al. (1994) examine a continuous-review $(Q, r)$ model and obtain approximate solutions of the optimal policy under the CAPM framework. For periodic-review inventory models, Inderfurth and Schefer (1996) analyze an order-up-to inventory policy and characterize the optimal reorder level for both the backorder case and the lost-sales case. The CAPM framework has also been used by Shan and Zhu (2013) and Rajagopalan (2013) to estimate the opportunity cost of capital tied-up in inventories. Despite using the CAPM approach, these studies do not incorporate randomness at the supply source, which is prevalent in modern supply chains (Bollapragada et al., 2004; Chopra and Sodhi, 2004) and a key focus of our paper.

Random capacity is a form of order quantity uncertainty without dependence on the quantity ordered, and multiple studies have examined its role in operations management. Ciarallo et al. (1994) consider an inventory model with random capacity and discover that the optimal inventory order decision is not affected by random capacity in the single-period case. Dada et al. (2007) consider purchasing from multiple suppliers with random capacity and find that the quantity ordered from each supplier depends on its reliability. Wang et al. (2010) examine the trade-off between dual sourcing and reducing randomness in supplier capacity and highlight factors influencing the trade-off. These studies do not consider the financial market risk of inventory decisions and limit the sources of capacity randomness to technical factors. Consequently, they do not recognize that random capacity can be the result of supplier business conditions associated with the financial market, a key aspect of our model. To the best of our knowledge, this paper is the first attempt to apply the CAPM framework to a company facing inventory decisions when both demand and capacity are random and correlated with the financial market.

We organize the rest of this paper as follows. Section 2 analyzes a newsvendor firm with random capacity under CAPM and explores the optimality conditions. Section 3 investigates the optimal inventory ordering decision and discusses comparative statics. Section 4 presents some numerical results. Section 5 summarizes our findings and provides managerial insights.

### 2.2 A newsvendor firm with random capacity under CAPM

We consider a company which faces random demand, random capacity, and a newsvendor-type inventory decision. The sequence of events for the company is as follows. First, the firm places an order of size $Q$; second, payment for goods is made after supplier delivery of the order at the beginning of a period; third, sales revenue (including revenue from salvaging unsold goods) is collected at the end of the period. Let random capacity $Y \sim F(Y)$ and random demand $Z \sim G(Z)$. Denote the unit selling price as $r$, the unit salvage value as $s$, the unit purchasing cost as $a$, and the risk-free interest rate as $r_f$. For clarity, we assume that $a$ is in beginning-of-period dollars while $r$ and $s$ are in end-of-period dollars.

We use the following notations:

$$
\begin{aligned}
Q &= \text{order quantity},\\
f(\cdot,\cdot), f(\cdot,\cdot,\cdot) &= \text{probability density function of two/three jointly distributed variables},\\
\Phi(\cdot),\ \phi(\cdot) &= \text{cumulative and density functions of the standard normal distribution},\\
\mu_X,\ \sigma_X &= \text{the expected value and the standard deviation of random variable } X,\\
&\quad \text{respectively},\\
\mathrm{Cov}(\cdot,\cdot) &= \text{the covariance operator},\\
\delta_{BC} &= \text{the correlation coefficient between random variables } B \text{ and } C,\\
a_B &= [A-\mu_B]/\sigma_B, \text{ where } B \text{ is a r.v., for example, } q_X = [Q-\mu_X]/\sigma_X,\\
r_M, r_f &= \text{the expected market return and the risk-free interest rate, respectively},\\
\Omega &= \text{the market price per unit of risk given by } \Omega = (r_M - r_f)/\sigma_M^2,\\
s_R &= \Omega \cdot \sigma_M = (r_M - r_f)/\sigma_M \text{ is Sharpe's (1964) ratio}.\\
M &= \text{the market return},\\
c_F &= \frac{r - a(1+r_f)}{r-s}, \text{ the newsvendor critical fractile}.
\end{aligned}
$$

In the following lines, we generalize the classic newsvendor model by using a CAPM framework. The CAPM penalizes *systematic risk*, defined as positive covariance of a firm's cash flow with the market return. To capture the systematic risk, we need to analyze the random cash flow of the newsvendor firm. For any given $Q$, the random end-of-period cash flow of the inventory investment $V(Q)$ is given by:

$$V(Q) = \begin{cases} rQ & \text{if } Y \geq Q \text{ and } Z \geq Q \\ rY & \text{if } Y < Q \text{ and } Z \geq Y \\ rQ - (r-s)(Q-Z) & \text{if } Y \geq Q \text{ and } Z < Q \\ rY - (r-s)(Y-Z) & \text{if } Y < Q \text{ and } Z < Y \end{cases} \tag{2.1}$$

We define an auxiliary function $U(Q) = \min\{Y, Q\}$. Letting $D_1(Q)$ be the end-of-period value of the negative cash flow (cash outlay) at the beginning of the period, we arrive at $D_1(Q) = -a(1+r_f)U(Q)$. The value of the random cash flow (or end-of-period realized random profit) at the end of period is $D(Q) = D_1(Q) + V(Q) = [r - a(1+r_f)] \cdot U(Q) - (r-s) \cdot [U(Q) - Z]^+$. In a simpler setting with no market correlation, Ciarallo et al. (1994) shows that the order quantity maximizing the expected profit $\mathbb{E}[D(Q)]$ for a random-capacity newsvendor, denoted by $Q_C$, is the same as that of the classical newsvendor model with unlimited capacity, namely $Q_C = G^{-1}\left(\frac{r-a(1+r_f)}{r-s}\right)$.

Following the approach of Kim and Chung (1989), our objective is maximizing the firm value after the inventory decision. Based on CAPM's additivity property (Thorstenson, 1988), this inventory decision is independent of other projects and products inside the firm. As a result, our objective is equivalent to maximizing the market valuation of the random cash flow associated with the inventory investment. In addition, this objective is equivalent to maximizing $S(Q)$, the increase in the current value of the firm as a consequence of the inventory investment project, where $Q$ is the decision variable. The increase in firm value $S(Q)$ based on the CAPM framework can be expressed as:

$$(1 + r_f)S(Q) = [\mathbb{E}(D(Q)) - \Omega\text{Cov}(D(Q), M)] \tag{2.2}$$

It is important to note that systematic risk enters the firm via correlation between the firm's cash flow and market return (as it can be observed from Equation 2.2). In other words, the correlations between demand/capacity with the market return channel systematic risk into the firm. To focus on the interplay of financial risk and capacity randomness, we assume independence between random demand and random capacity; such independence can be found when the supplier and the buyer firm are located in different countries and/or different industries. Moreover, we do not consider changing the selling price after observing the realized capacity level, since for long lead-time items, the prices are often announced before capacity randomness is realized (e.g. Walsh, 2008; Lowensohn, 2009). In the next section, we continue to analyze the ordering decision that maximizes the firm value increase ($S(Q)$).

## 2.3 Obtaining and analyzing the optimal ordering decision

We now consider the case that both the demand $Z$ and the capacity $Y$ are correlated with the market return $M$. Following previous studies (Anvari, 1987; Kim and Chung, 1989) and the fact that available capacity can often be approximated with a normal distribution, we assume that $Y$, $Z$, and $M$ are jointly normally distributed. For instance, a production system with unreliable parallel machines has a random available capacity that is binomially-distributed. In turn, such random capacity can be approximated by a normal distribution. We characterize the optimal order quantity $Q^*$ in Lemma 2.3.1. All proofs are in the Appendix.

*Lemma* 2.3.1. The optimal order quantity $Q^*$ is characterized by:

$$[1-\Phi(q_Y^*)]\,[c_F-\Phi(q_Z^*)-s_R\delta_{MZ}\phi(q_Z^*)]-s_R\delta_{MY}\phi(q_Y^*)\cdot[c_F-\Phi(q_Z^*)]=0 \qquad (2.3)$$

Lemma 2.3.1 indicates that the optimal ordering decision depends on both demand characteristics and capacity characteristics. We define $\Phi(q_Z^*)$ as the service level received by customers, which will be called *customer service level*, and $1-\Phi(q_Y^*)$ as the service level received by the firm from the supplier, which will be called *supplier service level*. Clearly, the optimal order quantity $Q^*$ relates to $q_Z^*=\frac{Q^*-\mu_Z}{\sigma_Z}$ and $q_Y^*=\frac{Q^*-\mu_Y}{\sigma_Y}$, impacting both customer service level and supplier service level.

We now introduce some notations that will be used later. Define

$$H_a(Q)=c_F-\Phi(q_Z)-s_R\delta_{MZ}\phi(q_Z)$$

and

$$H_b(Q)=c_F-\Phi(q_Z).$$

Let $Q_a$ characterized by $H_a(Q_a)=0$ be the *unlimited-capacity CAPM solution* (Anvari, 1987; Kim and Chung, 1989). Let

$$Q_b=\mu_Z+\Phi^{-1}\left(\frac{r-a(1+r_f)}{r-s}\right)\sigma_Z$$

denote the *classical newsvendor solution* and note that $H_b(Q_b)=0$. It can be shown that $H_a(Q)>0 \Leftrightarrow Q<Q_a$ and $H_b(Q)>0 \Leftrightarrow Q<Q_b$; conversely, $H_a(Q)<0 \Leftrightarrow Q>Q_a$ and $H_b(Q)<0 \Leftrightarrow Q>Q_b$. We shall call $\delta_{MZ}$ as the *market-demand correlation* and $\delta_{MY}$ as the *market-capacity correlation*. In Corollary 2.3.2, we investigate the relative position of $Q^*$ with respect to $Q_a$ and $Q_b$.

*Corollary* 2.3.2. The relative location of the optimal ordering decision ($Q^*$) with respect to the unlimited-capacity CAPM solution ($Q_a$) and the classical newsvendor solution ($Q_b$) can be characterized as follows:

(a) Under negative market-capacity correlation (i.e., $\delta_{MY} < 0$), the optimal solution $Q^*$ is located between $Q_a$ and $Q_b$; in this case, $Q^* \in (Q_a, Q_b)$ under positive market-demand correlation and $Q^* \in (Q_b, Q_a)$ under negative market-demand correlation.

(b) Under positive market-capacity correlation (i.e., $\delta_{MY} > 0$), the optimal solution $Q^*$ is not located between $Q_a$ and $Q_b$. In this case, $Q^* \notin (Q_a, Q_b)$ under positive market-demand correlation and $Q^* \notin (Q_b, Q_a)$ under negative market-demand correlation; if $\frac{\partial^2}{\partial Q^2} S(Q^*) < 0$, $Q^* < Q_a < Q_b$ under positive market-demand correlation, and $Q_b < Q_a < Q^*$ under negative market-demand correlation.

Under negative market-capacity correlation, we observe that the financial-market impact on the order quantity from the market-demand correlation is partially mitigated; under positive market-capacity correlation, the financial-market impact on the order quantity from the market-demand correlation is amplified; the intuitions behind these observations based on Corollary 2.3.2 are outlined in Subsection 3.1 after we present Proposition 2.3.5. We confirm the second-order condition $\frac{\partial^2}{\partial Q^2} S(Q^*) < 0$ during extensive numerical experiments reported in Section 5 and use this finding as an assumption in Propositions 2.3.3-2.3.7. We present comparative statics of the optimal ordering decision in the following subsections.

### *2.3.1 Impact of the financial market*

We begin by showing how the financial market risk appetite impacts the optimal ordering decision.

*Proposition* 2.3.3. As Sharpe's ratio ($s_R$) rises, the optimal ordering decision ($Q^*$) moves away from the classical newsvendor solution ($Q_b$).

When Sharpe's ratio rises, systematic risk becomes more undesirable, lifting the impact of financial market risk on firm value through demand uncertainty and capacity uncertainty. If $Q^* > Q_b$ or equivalently $\delta_{MZ} < 0$, as Sharpe's ratio rises, the optimal ordering decision increases to channel more *negative* systematic risk into the firm since the financial market penalizes systematic risk more heavily. In this case, a side-effect of a higher Sharpe's ratio is a higher customer service level, which puts pressure on the supplier service level. If $Q^* < Q_b$ or equivalently $\delta_{MZ} > 0$, the opposite is true since the firm avoids *positive* systematic risk more strongly as Sharpe's ratio rises. Managers should beware that their inventory decisions need to reflect the change in financial market conditions and prepare for the change in customer and supplier service levels.

Now we investigate how market-demand correlation impacts the optimal ordering decision.

*Proposition* 2.3.4. As market-demand correlation ($\delta_{MZ}$) becomes more positive, the optimal ordering decision ($Q^*$) decreases.

When market-demand correlation becomes more positive, the systematic risk introduced into the firm via market-demand correlation becomes more positive, since the impact of demand on the firm's cash flow is positive. Under this scenario, the optimal ordering decision decreases to avoid undertaking too much systematic risk since systematic risk becomes more undesirable. Moreover, lower order quantity drives customer service level lower and supplier service level higher. Managers should evaluate market-demand correlation carefully if they want to introduce a new product to the market, since the optimal order quantity can be impacted. It is also important to note that results similar to Propositions 2.3.3 and 2.3.4 can be found in unlimited-capacity CAPM newsvendor models (Anvari, 1987; Kim and Chung, 1989), but our study has extended these results to more general settings allowing for a finite capacity and a market-capacity correlation.

Now we investigate how market-capacity correlation impacts the optimal ordering decision.

*Proposition* 2.3.5. As market-capacity correlation ($\delta_{MY}$) becomes more positive, the optimal ordering decision ($Q^*$) moves away from the classical newsvendor solution ($Q_b$).

When market-capacity correlation becomes more positive, the systematic risk introduced into the firm via market-capacity correlation shifts towards being more positive. The outcome is that more negative market-capacity correlation may partially compensate for the impact of random capacity and pull the optimal ordering decision closer to the classical newsvendor solution. The intuition is that when both demand and capacity impact the firm's cash flow, we have that the end-of-period random cash flow $D(Q) = -[a(1 + r_f) - s]Y + (r - s)Z$, where the coefficient of capacity is opposite in sign and smaller in magnitude with respect to that of demand (i.e., $0 < a(1 + r_f) - s < r - s$). It follows that market-capacity correlation amplifies the impact of market-demand correlation if they have different signs (i.e., $\delta_{MY}\delta_{MZ} < 0$), which helps to explain the behavior described in Proposotion 2.3.5. Managers should note that market-capacity correlation impacts the optimal ordering decision and plan accordingly if market-capacity correlation changes.

### *2.3.2 Impact of capacity characteristics*

We now examine how capacity impacts the optimal ordering decision.

*Proposition* 2.3.6. As the mean capacity ($\mu_Y$) increases, the optimal ordering decision ($Q^*$) increases when $\delta_{MY}\delta_{MZ} > 0$ and decreases when $\delta_{MY}\delta_{MZ} < 0$.

Different from prior studies without firm-value considerations (Ciarallo et al., 1994), we find that capacity does influence order quantity. In Proposition 2.3.6, we have shown that the optimal ordering decision increases in the mean capacity only when $\delta_{MY}\delta_{MZ} > 0$ and find that an increase in mean capacity may reduce the optimal ordering quantity in some cases.

In particular, with a positive market-capacity correlation, a higher mean capacity brings more capacity-induced systematic risk to the firm. As a response to the increase in capacity-induced systematic risk, the optimal ordering decision moves towards both the unlimited-capacity CAPM solution ($Q_a$) and the classical newsvendor solution ($Q_b$) to avoid incurring more systematic risk. Conversely, with negative market-capacity correlation, the optimal ordering decision moves away from the classical newsvendor solution and towards the unlimited-capacity CAPM solution. This move leverages on the reduction in negative capacity-induced systematic risk when the mean capacity rises.

Managers should take caution that higher supplier capacity does not necessarily lead to a lower order quantity, as one may expect. For example, if the buyer firm switches to a new supplier with higher mean capacity *ceteris paribus*, the manager needs to increase the order quantity when the market-capacity correlation is identical in sign with the market-demand correlation.

### *2.3.3 Impact of product profitability*

We now examine how product profitability impacts the optimal ordering decision.

*Proposition* 2.3.7. We have the following results regarding the impact of product profitability:

(a) When $\delta_{MY} < 0$, the optimal ordering decision ($Q^*$) increases in the critical fractile;

(b) When $\delta_{MY} > 0$ and $Q^* < Q_{2b}$, the optimal ordering decision increases in the critical fractile;

(c) When $\delta_{MY} > 0$ and $Q^* > Q_{2b}$, the optimal ordering decision decreases in the critical fractile.

Note that $Q_{2b}$ is a critical threshold characterized by the equation $1-\Phi\left(\frac{Q_{2b}-\mu_Y}{\sigma_Y}\right)-s_R\delta_{MY}\phi\left(\frac{Q_{2b}-\mu_Y}{\sigma_Y}\right)=0$. We find that under most circumstances, the optimal ordering decision increases in the critical fractile $c_F$ since improved profitability encourages a higher order quantity, increasing customer service level while reducing supplier service level. However, managers should be careful that improved profitability may mean a smaller order quantity when the impact of random capacity is extremely strong (i.e., $Q^* > Q_{2b}$) and thus avoid always expanding the order quantity whenever product profitability improves.

## 2.4 Numerical analysis

In this section, we explore the impact of system parameters further. We present results from numerical experiments with $\rho = \mu_Z/\mu_Y = 0.6/1.0$, $a = 50/90$ to account for different levels of capacity utilization and product profitability. We assume $\mu_Z = 10,000$, $\sigma_Z = 3,000$, $\{\mu_Y, \sigma_Y\} = \{16,667,\ 5,000\}$ or $\{10,000,\ 3,000\}$, $r = 100$, $s = 10$, and that both the capacity and the demand are normally distributed. The return of large-company stocks (i.e., the Standard & Poor's 500 Index) is used as a proxy for the market return, and the return of 10-year U.S. Treasury bonds is used as the risk-free interest rate. We obtain $r_M = 11.8\%$ and $\sigma_M = 20.3\%$ from Morningstar Inc. (2012, p.32) and $r_f = 3.6\%$ from Morningstar Inc. (2012, p.53). Define margin as $\beta = \frac{r-a(1+r_f)}{r}$. We focus on the difference in firm value and order quantity between the the optimal inventory decision $Q^*$ and the classical newsvendor

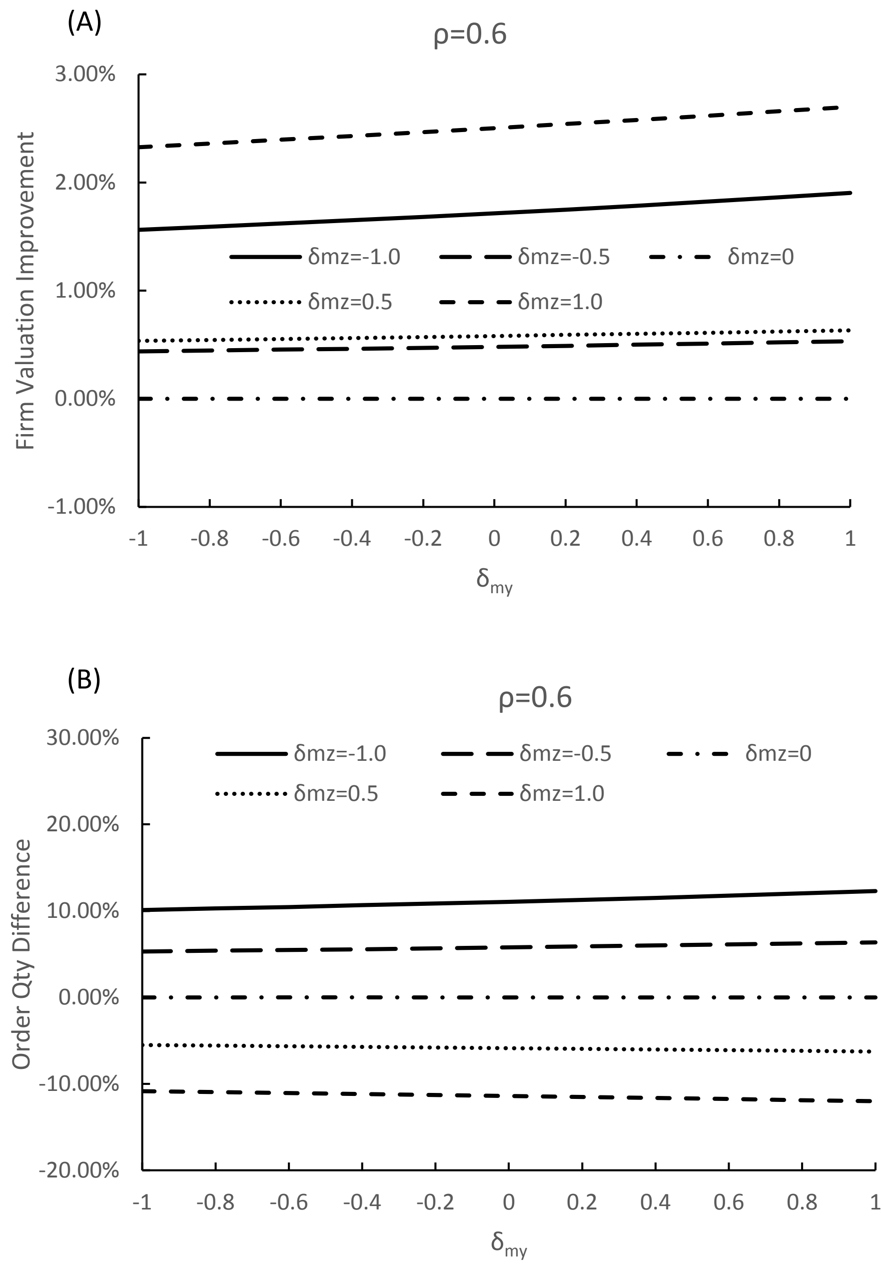



Figure 2.1: High Margin Scenario ($a = 50$) with Low Capacity Utilization ($\rho = 0.6$)

solution $Q_b$. We denote the firm value difference as

$$\Delta S = \frac{S(Q^*) - S(Q_b)}{S(Q_b)}$$

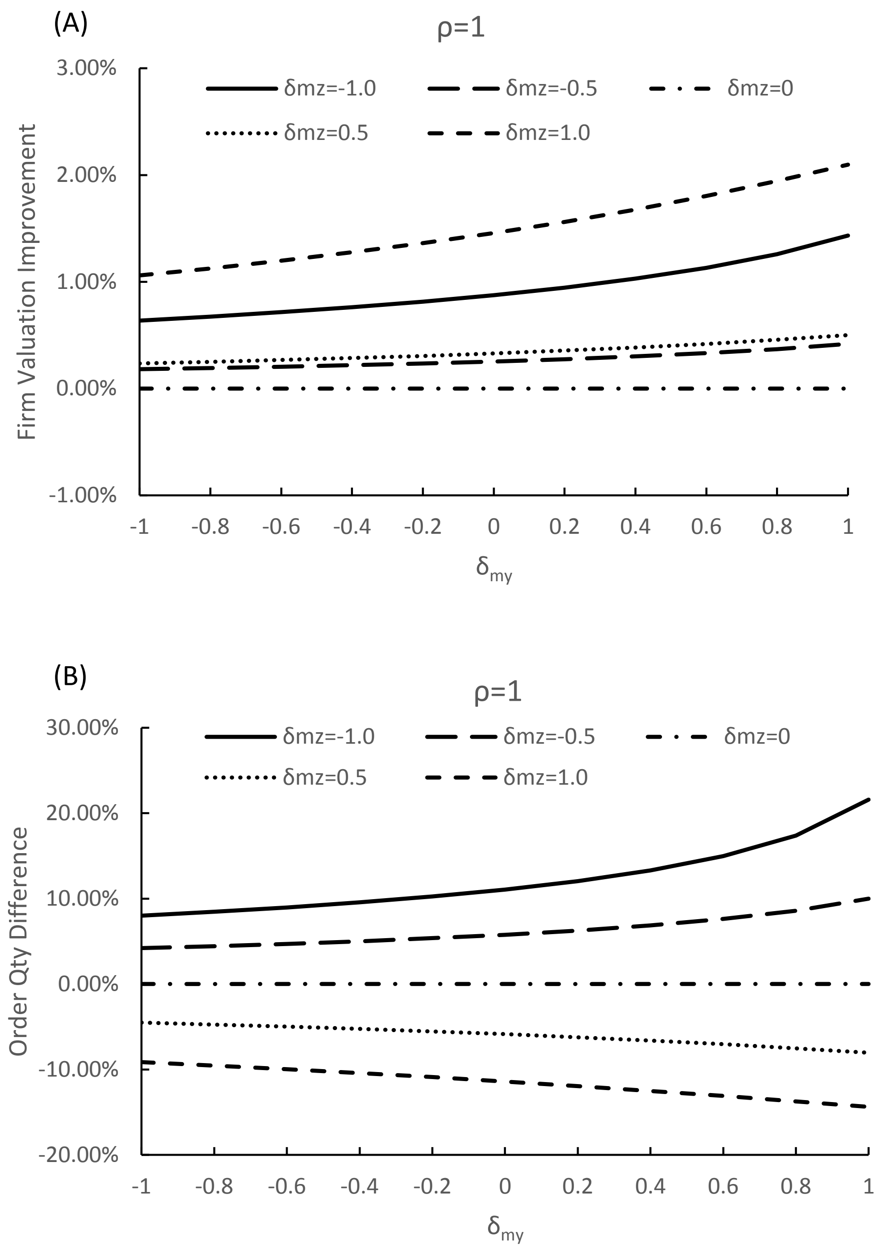
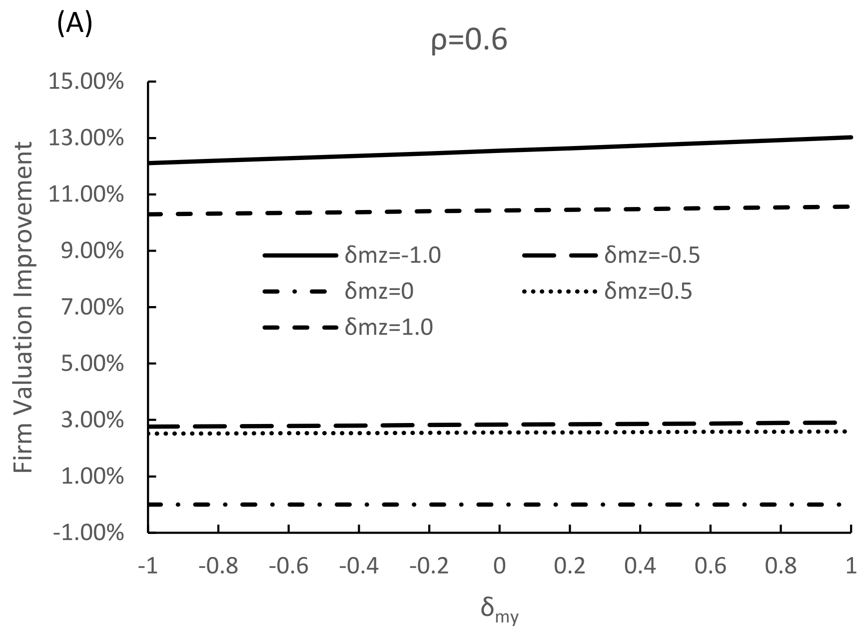



Figure 2.2: High Margin Scenario ($a = 50$) with High Capacity Utilization ($\rho = 1.0$)

and order quantity difference as

$$\Delta Q = \frac{Q^* - Q_b}{Q_b}.$$



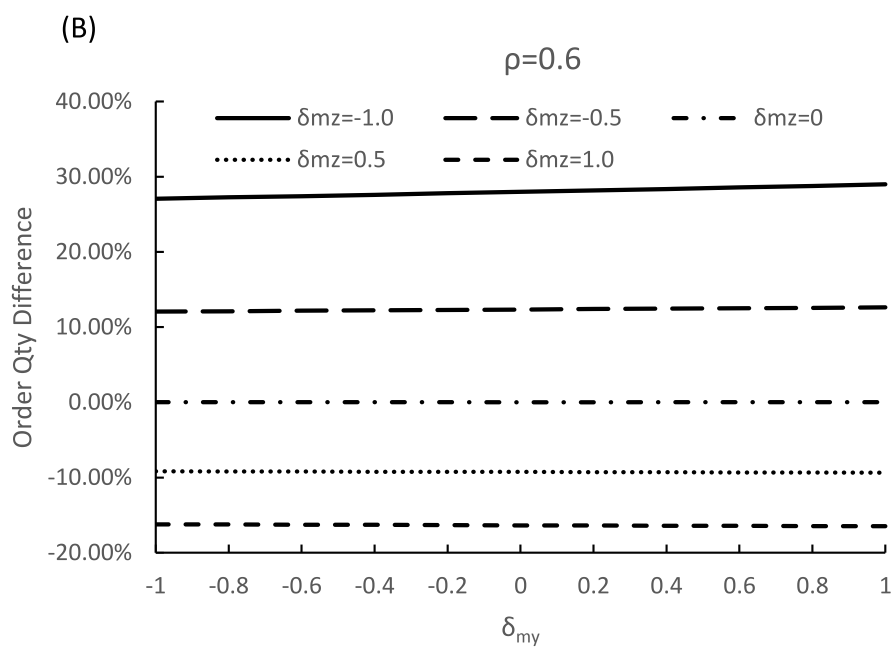




Figure 2.3: Low Margin Scenario ($a = 90$) with Low Capacity Utilization ($\rho = 0.6$)

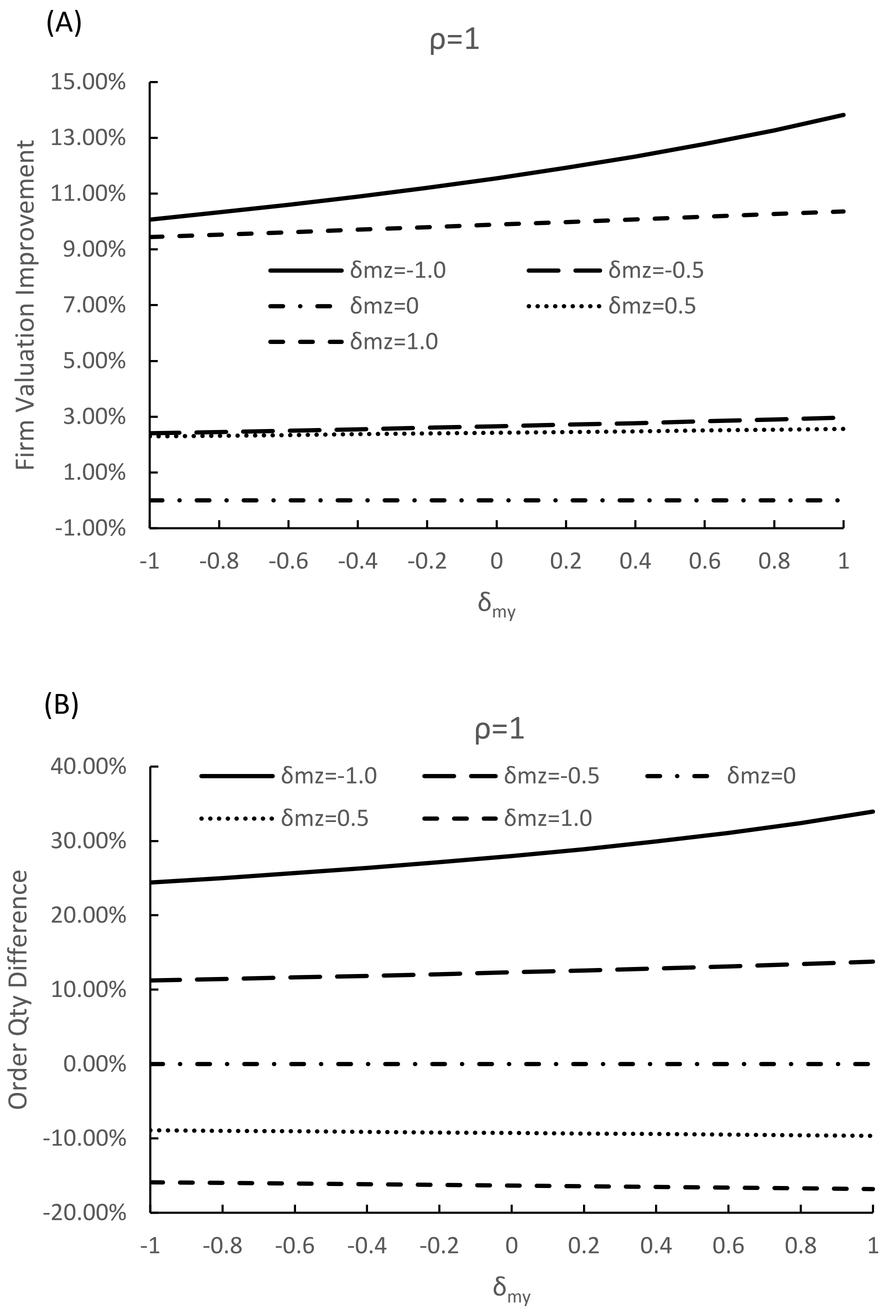



Figure 2.4: Low Margin Scenario ($a = 90$) with High Capacity Utilization ($\rho = 1.0$)

The numerical results are outlined in Figures 2.1 to 2.4, and in addition to confirming our analytical results, we have three observations based on the numerical results:

*Observation* 1. The percentage difference between the optimal inventory decision and the classical newsvendor solution ($\Delta Q$) increases in the profit margin.

This increase is caused by a higher stake in unfulfilled demand in the high margin scenario. Therefore, managers are advised to be more careful about financial market risks for higher-profit products.

*Observation* 2. The sensitivity of valuation difference ($\Delta S$) and quantity difference ($\Delta Q$) to the market-capacity correlation ($\delta_{MY}$) increases in the capacity utilization ($\rho$).

This observation can be explained by the scarcity in capacity when capacity utilization is high. Managers should pay more attention to supply-side risk under this situation due to its impact on order quantity and firm value.

*Observation* 3. The larger the strength of market-demand correlation, the more sensitive is the valuation difference ($\Delta S$) to the market-capacity correlation, which is amplified under negative market-demand correlation.

The intuition is that a stronger market-demand correlation exposes the firm to a higher magnitude of financial risk, and thus the market-capacity correlation becomes more critical in neutralizing this financial risk. With a negative market-demand correlation, the optimal ordering decision is higher than the classical newsvendor solution. As a result, the random capacity and the market-capacity correlation become more influential as the optimal ordering decision rises and capacity becomes tighter. Therefore, managers should pay more attention to capacity risks when market-demand correlation is highly negative.

### 2.5 Process improvement on random capacity

In this section, we investigate the impact of random capacity process improvement on firm value when the random capacity is uncorrelated with market returns and the random capacity is generally distributed. Although process improvement is believed to increase the firm value (Hendricks and Singhal, 1996; Keen, 1997) and impact capital adequacy (Mizgier et al., 2015), it is not clear how and when the firm value is maximized on the financial market via process improvement. To the best of our knowledge, this essay is the first one to analyze the firm-value impact of improving a generally-distributed random capacity using the CAPM framework.

As previously mentioned, we consider the case where the demand $Z$ is correlated with the market return $M$ under the assumption that $Z$ and $M$ are jointly normally distributed. In other words, $\text{Cov}(Z, M) \neq 0$, $Z \sim N(\mu_Z, \sigma_Z^2)$, and $M \sim N(r_M, \sigma_M^2)$. Since the capacity randomness is due to technical failures and other exogenous factors, $Y$ and $M$ are independent, namely $\text{Cov}(Y, M) = 0$. We assume that $f(Y)$ is continuous and has support $(Y_{min}, Y_{max})$.

The firm-value-maximizing order decision is characterized in Lemma 2.5.1.

*Lemma* 2.5.1. The optimal order quantity $Q^* = \mu_z + q_Z^* \sigma_Z$ is characterized by:

$$\Phi(q_Z^*) + s_R\, \delta_{MZ}\, \phi(q_Z^*) = \frac{r - a(1 + r_f) + d}{r - s + d} \tag{2.4}$$

Lemma 2.5.1 provides a necessary condition of the optimal solution, and we continue to analyze the second-order condition in Lemma 2.5.2.

*Lemma* 2.5.2. $Q^*$ satisfies the second-order condition (SOC): $\frac{\partial^2}{\partial Q^2} S(Q^*) < 0$ when $Q^* < Y_{max}$.

Having shown that $Q^*$ satisfies SOC in Lemma 2.5.2, we prove the optimality of $Q^*$ in Theorem 2.5.3.

*Theorem* 2.5.3. $Q^*$ characterized by Equation 2.4 is the optimal order quantity.

(a) $Q^*$ is the unique optimal order quantity when $Q^* < Y_{max}$.

(b) $Q^*$ remains optimal but not uniquely optimal when $Q^* \geq Y_{max}$ since any order quantity no less than $Y_{max}$ is optimal.

We focus on the discussion of Theorem 2.5.3(a) and assume $Q^* < Y_{max}$ hereafter in this essay since Theorem 2.5.3(b) is a trivial case and only relevant when the capacity is extremely scarce. Theorem 2.5.3(a) coincides with previous results in the classical newsvendor model under CAPM (Anvari, 1987; Kim and Chung, 1989), suggesting that the optimal order quantity under CAPM depends on the correlation between demand and market return in addition to parameters already incorporated in the classic newsvendor model. It follows that the financial market can impact ordering decisions via the random demand since demand is correlated with market return. Moreover, the fact that random capacity has no impact on the optimal order quantity is analogous to similar findings in the classical newsvendor model (Ciarallo et al., 1994), suggesting that the financial market cannot impact ordering decisions via the random capacity under this scenario, since the random capacity is not correlated with market return.

Let

$$c_F = \frac{r - a(1 + r_f) + d}{r - s + d}$$

be the critical fractile (incorporating interest cost) and $Q_C = G^{-1}(c_F)$ be the classical newsvendor solution, noting that $Q^* = Q_C$ when $\delta_{MZ} = 0$. It is easy to show that $Q^* > Q_C$ when $\delta_{MZ} < 0$ and $Q^* < Q_C$ when $\delta_{MZ} > 0$. We analyze the differences between the ordering decision that maximizes firm value and that maximizes the expected profit in Corollary 2.5.4.

*Corollary* 2.5.4. We have the following results:

(a) $Q^*$ moves away from $Q_C$ as $s_R$ increases.

(b) $Q^*$ moves away from $Q_C$ as $|\delta_{MZ}|$ increases.

(c) As $\sigma_Z$ increases, both $Q^*$ and $Q_C$ move away from $\mu_Z$, and $|Q^* - Q_C|$ increases.

(d) As $\mu_Z$ increases, both $Q^*$ and $Q_C$ increase with $dQ^*/d\mu_Z = dQ_C/d\mu_Z = 1$, and $|Q^* - Q_C|$ stays the same.

(e) As $c_F$ increases, both $Q^*$ and $Q_C$ increase.

Based on Corollary 2.5.4 (a), we find that since a higher Sharpe's ratio ($s_R$) means a higher penalty by the financial market on systematic risk, $Q^*$ is pushed away from $Q_C$ to optimize the firm's risk profile. Based on Corollary 2.5.4 (b), a larger association between demand and market return makes the firm more susceptible to risk in the financial market; thus deviation from the classical newsvendor solution is desired to reduce the risk as $|\delta_{MZ}|$ increases. It follows that when systematic risk introduced through demand is higher ($s_R$ and $|\delta_{MZ}|$ are higher), managers need to pay special attention to their order quantity to maximize firm value.

We also find from Corollary 2.5.4 (c) that as demand risk $\sigma_Z$ increases, both $Q^*$ and $Q_C$ move away from $\mu_Z$; despite both moving in the same direction, the relative difference increases in $\sigma_Z$, suggesting higher importance to account for financial risk when demand variability rises. We also discover that the mean demand $\mu_Z$ has no impact on the relative difference between $Q^*$ and $Q_C$ based on Corollary 2.5.4 (d).

As the newsvendor critical fractile $c_F$ increases, both $Q^*$ and $Q_C$ rise due to increased profitability based on Corollary 2.5.4 (e), while their relative difference remains the same due to an unchanged demand profile. We acknowledge that Kim and Chung (1989) obtained results similar to Corollary 2.5.4 (b), (c) and (e) for the classical newsvendor model under CAPM. However, different from Kim and Chung (1989), we consider a more general setting by incorporating a random capacity limit and allowing $\delta_{MZ}$ to be negative, broadening the applicability of our results.

### *2.5.1 The role of capacity process improvement*

In this section, we investigate the impact of capacity process improvement on firm value. We first develop several properties of capacity process improvement under the classic newsvendor model with random capacity, and then investigate if these properties also apply under the CAPM newsvendor model with random capacity. In Sections 3 and 4, we only consider the case where capacity process improvement matters (i.e., $Q^* > Y_{min}$). Let

$$\begin{aligned} P(Y) = [r - a(1 + r_f)]Y - (r - s)\int_0^Y (Y - Z)g(Z)dZ - d\int_Y^{+\infty}(Z - Y)g(Z)dZ \\ - (r - s + d)\delta_{MZ}s_R\Phi\left(\frac{Y - \mu_Z}{\sigma_Z}\right) \end{aligned} \tag{2.5}$$

Also denote

$$\overline{P}(Y) = \begin{cases} P(Y) & \text{if } Y < Q^* \\ P(Q^*) & \text{if } Y \geq Q^* \end{cases}$$

The expression of the expected valuation increase of the newsvendor firm with random capacity can be rewritten as

$$S(Q^*) = (1 + r_f)^{-1}\left\{P(Q^*)[1 - F(Q^*)] + \int_0^{Q^*} P(Y)f(Y)dy\right\} \tag{2.6}$$

We begin our analysis by examining the role of capacity expansion without changing the shape of the p.d.f. of the random capacity, meaning that the mean capacity improves, but the variability of capacity remains the same. We name this type of capacity process improvement as *mean-capacity improvement.*

We also examine the benefits of process improvement by rescaling the p.d.f. of the random capacity towards the mean. We name this type of capacity process improvement as *capacity-variance reduction*. Based on the transformation $\widetilde{f}(Y) = bf\left[\mu_Y + b(Y - \mu_Y)\right]$ where $b > 0$, the p.d.f. of the random capacity shrinks toward its mean when $b > 1$, remains unchanged when $b = 1$, and expands when $0 < b < 1$. Recall that $S(Q^*)$ is the firm valuation increase without capacity-variance reduction. We denote $\widetilde{S}(Q^*) = \int_0^{+\infty} \overline{P}(Y)\widetilde{f}(Y)dY$ as the firm valuation increase after capacity-variance reduction. We explore how the financial market impacts the role of process improvement under the CAPM framework in Proposition 2.5.5.

*Proposition* 2.5.5. We have that

(a) With mean-capacity improvement, the firm value increases.

(b) If $\delta_{MZ} \in [0, 1]$, the following are true for a firm with random-capacity:

  (i) The firm value increase in (a) has diminishing returns and is bounded by the firm value under unlimited capacity.

  (ii) With capacity-variance reduction, the firm value increases.

  (iii) The firm value increase in (ii) has diminishing returns and is bounded by the firm value under deterministic capacity.

(c) If $\delta_{MZ} \in [-1, 0)$, under the sufficient condition $Y_{min} \geq \mu_Z + \frac{\sigma_Z}{\delta_{MZ} s_R}$, (i)-(iii) in part (b) are true.

To illustrate the implications of Proposition 2.5.5, if the random capacity is normally distributed with mean $\mu$ and variance $\sigma^2$, then either increasing $\mu$ or reducing $\sigma$ (while keeping the other parameter unchanged) increases the expected profit with diminishing returns and bounded by the expected profit under unlimited capacity.

Proposition 2.5.5(a) and (b)(i) demonstrate the benefits of capacity expansions but also caution its diminishing returns. It is worth noting that the condition in

Proposition 2.5.5(c) is independent of $c_F$, meaning that the only requirement is that the firm's risk profile satisfies certain properties. Focusing on the case when $\delta_{MZ} \in [-1, 0)$, we find that when $\sigma_Z$ increases, $\delta_{MZ}$ increases, $\mu_Z$ decreases, and $s_R$ decreases, it becomes easier to satisfy the sufficient conditions in Proposition 2.5.5(c). Although Proposition 1(c) may not always be satisfied, especially when $|\delta_{MZ}|$ is large, we show in Section 2.5.2 that the results of Proposition 2.5.5 may hold even when the sufficient condition in Proposition 2.5.5(c) is not satisfied.

### *2.5.2 Factors impacting process improvement*

In this section, we focus on the impact of system parameters on the relationship between process improvement and firm value with analytical results complemented by numerical studies. Defining capacity scarceness as the ratio between mean demand and mean capacity (i.e., $\rho = \mu_Z/\mu_Y$), we present numerical experiments with $\rho = 0.1$ or 1.0, $r = 100$, $a = 50$ or 90, $s = 0$, 40 or 80 (we only consider cases where $s < a$) and $d = 0$ to account for different levels of capacity scarceness and product profitability. We assume $\mu_Z = 10,000$, $\sigma_Z = 1,000$, 2,000 or 3,000, $\mu_Y = 100$,000 or 10,000, and $\sigma_Y/\mu_Y$=0.1, 0.2, or 0.3. We assume that both the capacity and the demand are normally distributed. We use the return of large-company stocks (e.g. the S&P 500 Index) as a proxy for the market return and the return of 10-year U.S. Treasury bonds as the risk-free interest rate. We obtain $r_M = 11.8\%$ and $\sigma_M = 20.3\%$ from Morningstar Inc. (2012, p.32) and $r_f = 3.6\%$ from Morningstar Inc. (2012, p.53). We focus on the difference in firm value when the supply source undergoes process improvement. The numerical results are outlined in Figures 1-3.

We focus on the following questions in the analysis:

1. What are the roles of the market-demand correlation ($\delta_{MZ}$) and Sharpe's rartio ($s_R$) in capacity process improvement?

2. What is the role of the capacity scarceness ($\rho$) in capacity process improvement?

3. What is the role of the critical fractile ($c_F$) in capacity process improvement?

#### *2.5.2.1 Impact of financial risk*

In Figure 2.5, we see that as the ratio $\sigma_Y/\mu_Y$ decreases from 0.3 to 0.1, the valuation difference $S(Q)$ increases; however, the curve with $\delta_{MZ} = -1$ increases faster than the curve with $\delta_{MZ} = 0$, which in turn increases faster than the curve with $\delta_{MZ} = 1$. We also notice diminishing returns in reducing capacity variability (see Figure 2.5 for example), coinciding with Proposition 2.5.5. These observations align with Corollary 2.5.6.

*Corollary* 2.5.6. We have the following results about the impact of financial risk:

(a) Regarding mean-capacity improvement:

  (i) When the market-demand correlation ($\delta_{MZ}$) increases, mean-capacity improvement is less beneficial to the firm's market valuation.

  (ii) When Sharpe's ratio ($s_R$) increases, mean-capacity improvement is more beneficial to the firm's market valuation under negative market-demand correlation and less beneficial under positive market-demand correlation.

(b) Regarding capacity-variance reduction:

  (i) When the market-demand correlation ($\delta_{MZ}$) increases, capacity-variance reduction is less beneficial to the firm's market valuation if $Q^* \leq \mu_Y$.

  (ii) When Sharpe's ratio ($s_R$) increases, capacity-variance reduction is more beneficial to the firm's market valuation under negative market-demand correlation and less beneficial under positive market-demand correlation, if $Q^* \leq \mu_Y$.

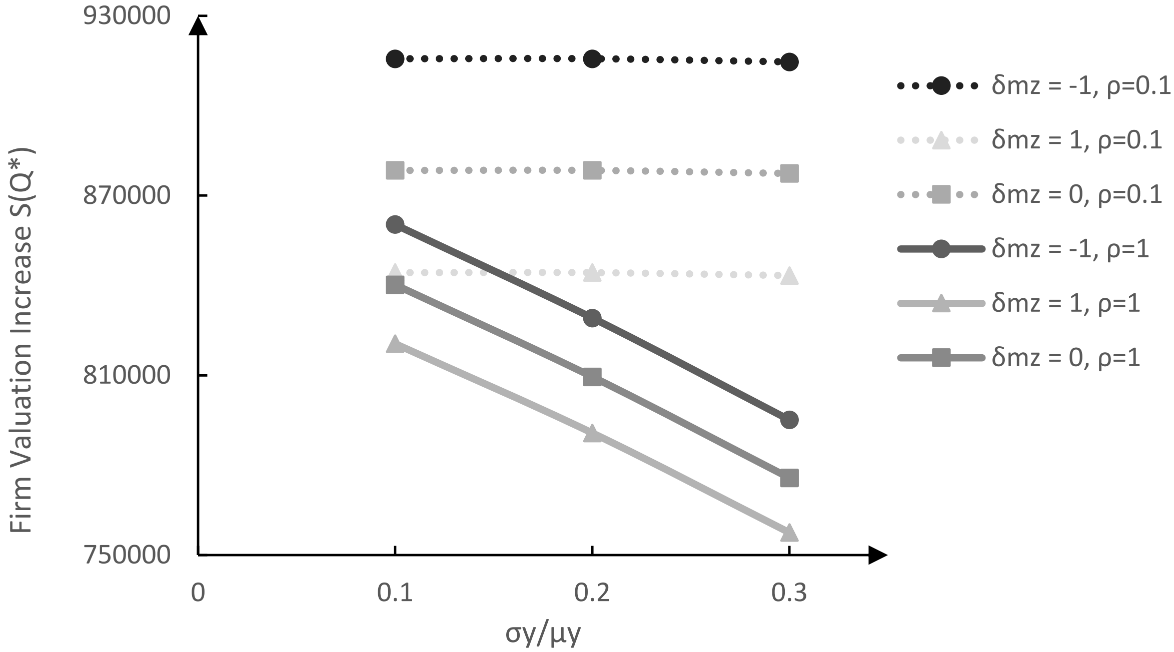



Figure 2.5: The Role of $\delta_{MZ}$ in Firm Valuation ($a = 90$, $s = 80$, $\rho = 1$)

We attribute this observation to the negative systematic risk in this case, which makes ordering additional units more desirable, since doing so brings market valuation increases to the firm. Firms should beware that their process improvement efforts may not bring the anticipated benefits if their demand is positively correlated with market returns. On the contrary, given the same level of investment in process improvement, firms with demand negatively correlated with market returns enjoy greater firm valuation increases compared to firms with demand positively correlated with market returns. It follows that firms are advised to understand their demand characteristics prior to investing in capacity improvement projects, since investments in these projects can only be justified after having reasonable expectations for their firm value benefits.

#### *2.5.2.2 Impact of capacity scarceness*

When $\rho$ is high, capacity process improvement matters more. In Figure 2.6, we can see that when capacity scarceness $\rho = 1$, the increase in valuation difference $S(Q)$, as $\sigma_Y/\mu_Y$ decreases from 0.3 to 0.1, is larger than that when $\rho = 0.1$. We also notice diminishing returns in process improvement in the mean capacity by comparing curve pairs with the same $\delta_{MZ}$ value (and different values of $\rho$) in Figure 2.6. These observations align with Corollary 2.5.7.

*Corollary* 2.5.7. When the capacity is more scarce, both mean-capacity improvement and capacity-variance reduction are more beneficial.

According to Corollaries 2.5.6 and 2.5.7, firms should devote more efforts to capacity improvement projects when (i) their capacity is tight and (ii) the market-demand correlation is highly negative, since the potential benefits of capacity process improvement is high when both conditions are satisfied. However, due to the diminishing returns in capacity process improvement, managers should monitor the benefits and avoid over-investing in such improvement.

#### *2.5.2.3 Impact of the critical fractile*

When the critical fractile $c_F$ is high, capacity process improvement matters more, since not receiving a unit of product costs more. Taking $c_F = 0.897$ for the parameter setting in Figure 2.7 and $c_F = 0.338$ for the parameter setting in Figure 2.6 as examples, we observe that the impact of process improvement is larger in Figure 2.7, by comparing curve pairs with the same $\delta_{MZ}$ value in each figure. This observation aligns with Corollary 2.5.8.

*Corollary* 2.5.8. With higher critical fractile, both mean-capacity process improvement and capacity-variance reduction become more beneficial.

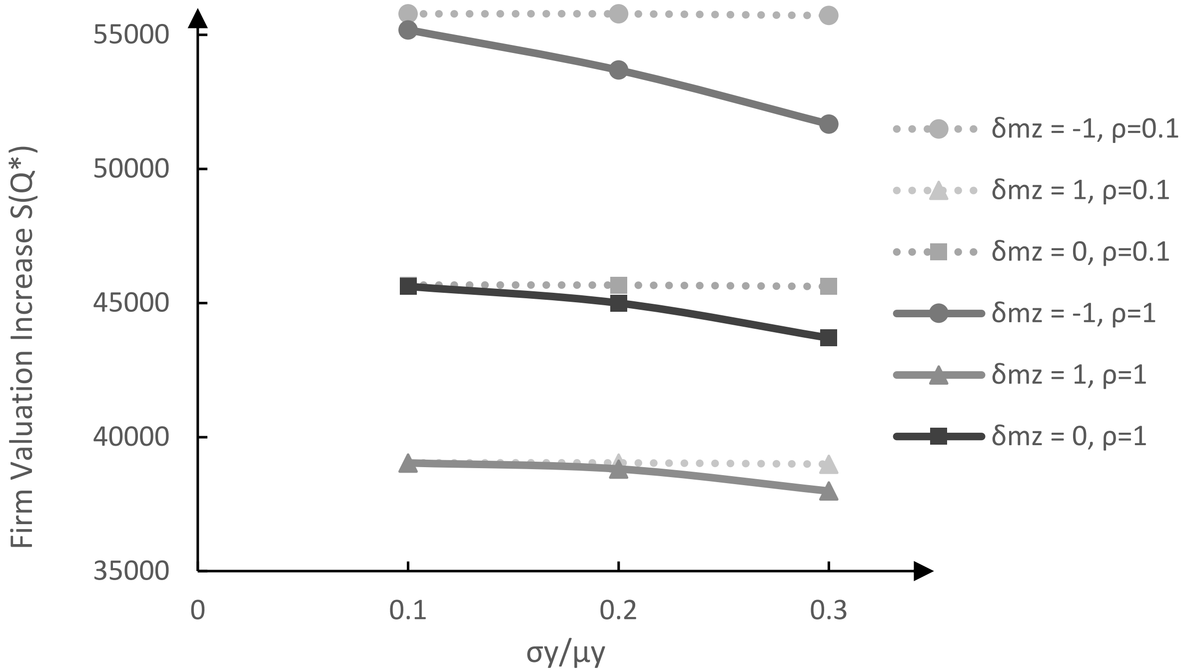



Figure 2.6: The Role of $\rho$ in Firm Valuation ($a = 90$, $s = 80$, $\rho = 0.1/1$)

Corollaries 2.5.7 and 2.5.8 coincide with empirical findings that small and medium-sized firms, which typically have lower margins and lower capacity cost, focus on reactive measures such as overcapacity and safety stock in production to manage supply chain risk, while large firms, typically with higher margins and higher capacity cost, focus on preventive measures such as strategic supplier development to improve their supply process (Thun et al., 2011).

## 2.6 Summary and conclusions

In this essay, we analyze a firm with random capacity. We use the CAPM framework to study how the financial market risk impacts the firm's optimal inventory decisions. Our results lead to several major findings.

First, we highlight that capacity randomness may influence the optimal order quantity, sometimes significantly, in contrast to previous models that do not incor-

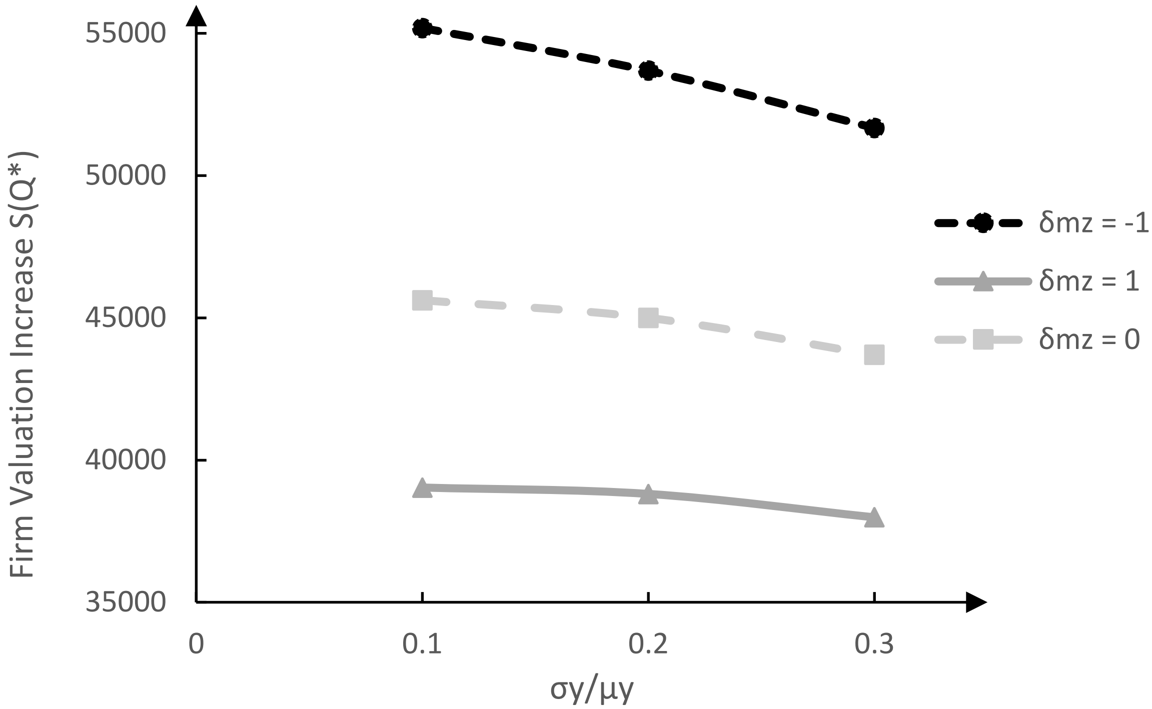



Figure 2.7: The Role of $c_F$ in Firm Valuation ($a = 10$, $s = 0$, $\rho = 0.1/1$)

porate financial market risk (e.g. Ciarallo et al., 1994) or do not include supply randomness (e.g. Anvari, 1987; Kim and Chung, 1989). Our first finding relates to the case in which the random capacity is negatively correlated with market return (e.g., many industrial products): the random capacity partially offsets the effect of demand randomness, resulting in an optimal order quantity similar (but not identical) to the classical newsvendor solution. In this case, the random capacity serves as a *risk-neutralizer* since its negative correlation with market return mitigates the systematic risk. We also find that when the random capacity is increasingly positively correlated with market return (e.g., contra-business-cycle products), the optimal order quantity moves away from the classical newsvendor solution. In this case, the random capacity serves as a *risk-amplifier* since its positive correlation with market return amplifies the systematic risk.

Second, we find that a higher correlation between random demand and market return is associated with a higher impact of the correlation between random capacity and market return. Thus, managers should pay more attention to supply-side risk when demand is highly correlated with market return.

Third, we show that when capacity utilization is tight or product margin is high, the random capacity may impact the order quantity considerably. Managers are also advised to incorporate random capacity and its correlation with the market return when product margin is low and capacity utilization is high, avoiding excessive exposure to systematic risk.

Fourth, we discover that an increase in average capacity does not necessarily lead to an increase in the optimal ordering quantity due to avoidance of capacity-induced systematic risk. We suggest that managers should carefully evaluate different scenarios instead of relying on rules of thumb to increase order quantity whenever average capacity improves.

Fifth, we highlight that the random capacity does not influence the optimal order quantity, coinciding with previous studies that do not incorporate financial risk (e.g. Ciarallo et al., 1994) or do not consider supply risk (e.g. Anvari, 1987; Kim and Chung, 1989). However, although CAPM builds on the risk-averseness of investors and demonstrates a trade-off between risk and returns, our finding differs from that of Wu et al. (2013), where capacity uncertainty decreases the order quantity for a risk-averse newsvendor. Moreover, we demonstrate that the correlation between demand and market return does impact the benefit of process improvement. In general, we find that higher correlation between demand and market return is associated with higher impact of capacity on profit. In particular, we discover that when demand is negatively associated with market return, capacity process improvement can be more beneficial than that when demand is not correlated or positively correlated with market return. We also find that both higher capacity scarceness and higher profitability increase the benefit of process improvement. Our findings can help managers rationalize investment in process improvement activities by predicting the increase in firm value brought about by process improvement. These findings may also guide managers in choosing among suppliers (if the supply source is external) based on each supplier's contribution to firm value.

Future research may extend the CAPM framework to other strategic and operational decisions, such as acquiring suppliers with random capacity and evaluating capacity investments with consideration of supply disruptions.

# 3. SUPPLY-CHAIN MANAGEMENT UNDER THE CONDITIONAL VALUE-AT-RISK CRITERION

## 3.1 Introduction

Recent economic studies show that the majority of individuals are risk averse (Halek and Eisenhauer, 2001; Holt and Laury, 2002; Eckel and Grossman, 2002; Harrison et al., 2007; Dave et al., 2010) and business managers are no exception (Amihud and Lev, 1981; Schweitzer and Cachon, 2000; Haigh and List, 2005; Harrison et al., 2009; Arreola-Risa and Keys, 2013). However, most of the published research on stochastic supply chains either explicitly assumes decision makers are risk-neutral or implicitly does so by focusing on minimization of expected cost. In this essay, we study risk-averse decision makers who manage stochastic and capacitated supply chains. The impetus for our research project stems from a consulting engagement with one of the five largest oil and gas companies in the world. For confidentiality reasons, the company will be called *Company A*.

Consider a supply chain which makes a single but very expensive item, say in the hundreds of thousands of dollars. The item's demand and production rates are random with respective averages $\lambda$ and $\mu$. To keep the analysis tractable and at the same time maximize research insight, we will follow related supply chain research (e.g. Zipkin, 1986; Arreola-Risa, 1996) and will model demand and production as Poisson processes. To smooth the random interaction of demand and production, an inventory of the item may be desirable. The inventory holding cost rate per unit is $h$ and because the ordering cost is negligible, the inventory is to be managed by a base-stock policy with parameter $B$, whose value is a *decision variable*. Demands that arrive when the inventory is temporarily depleted are back-ordered at a cost rate of $b$ per unit. Cost in a period is equal to the sum of inventory holding cost and demand back-ordering cost in the period. Note that as a consequence of randomness in both demand and production rates, cost in a period is a random variable as well. Because managers are typically evaluated on their short-term performance (Narayanan, 1985; Laverty, 1996; Marginson and McAulay, 2008), for example on an annual basis, without loss of generality, we assume in this paper a period is one year and the unit of analysis is annual cost. From this point on, the just-described supply chain will be called the supply chain under study.

The supply chain manager is risk-averse and due to the order of magnitude of annual cost, say in the hundreds of millions of dollars, he/she is interested in minimizing the following risk criterion: the average cost of all possible scenarios in the top $(1 - \beta)$ percentile of the annual cost probability distribution. In the financial risk literature, this criterion is known as *Conditional Value at Risk* and is usually denoted by $CVaR_\beta$. The parameter $\beta$ reflects the decision maker's *risk sensitivity* (the higher the $\beta$, the higher the risk sensitivity) and $0 < \beta < 1$. As we will see, $CVaR_\beta$ is very intuitive and easy to use; interestingly enough, the supply-chain manager in Company A wanted to use $CVaR_\beta$ based on his business experience and without knowing it was a formal criterion in the financial risk literature. At the same time, according to Rockafellar and Uryasev (2002), $CVaR_\beta$ has many desirable theoretical properties (positive homogeneity, translation-invariance, monotonicity, sub-additivity, law-invariance, and co-monotonic additivity). These properties make $CVaR_\beta$ a spectral risk measure, meaning that it is an excellent and coherent representation of subjective risk aversion. From this point on, the above described risk-averse supply-chain manager will simply be called the manager.

In the supply chain under study, the value of $B$ is the manager's decision variable. Obviously, different $B$ values yield different annual cost probability distributions.

This paper addresses the following research questions:

1. What is the value of $B$ whose associated annual cost probability distribution has the minimum $CVaR_\beta$? This value will be denoted by $B^*$.

2. Under which conditions (if any) would a stockless operation be optimal (i.e., $B^*=0$)?

3. What should be the manager's optimal adjustment (if any) to $B^*$ as the values of the parameters $\lambda$, $\mu$, $\beta$, $h$ and $b$ change?

4. Let $B^*_{EC}$ denote the base-stock value that minimizes *expected* annual cost. How does the value of $B^*$ compare to $B^*_{EC}$ and what would be the penalty (if any) in terms of $CVaR_\beta$ for using $B^*_{EC}$ instead of $B^*$?

The remainder of this paper is organized into five sections. Section 2 contains the mathematical definition of the first research question and a brief literature review. Section 3 establishes that an exact expression of $B^*$ is intractable, and derives an easy-to-use approximation of $B^*$. Section 3 also presents a simulation study conducted to test the accuracy of the easy-to-use approximation of $B^*$ and identifies conditions for the optimality of a stockless operation, answering the second research question. Section 4 answers the third research question by means of comparative statics. Section 5 deals with the fourth research question. Section 6 summarizes our research insights and proposes some ideas for future research on managing stochastic and capacitated supply chains under risk aversion.

## 3.2 Definitions and literature review

Let $B \equiv S + 1$. Since demand is a Poisson process, every arriving demand will be for one unit of product and hence $S$ could be interpreted as the reorder point.

For mathematical convenience, we will find $B^*$ by finding first the optimal value of $S$, from now on denoted by $S^*$.

Let $K(S)$ denote the random annual cost as a function of $S$, $p_{K(S)}(\cdot)$ denote the probability distribution of $K(S)$, and let $\eta_\beta$ be the lowest amount such that with probability $\beta$, the annual cost will not exceed $\eta_\beta$. For brevity, we use $\zeta_\beta(S)$ to denote $CVaR_\beta(S)$. For a given $S$, let

$$\Psi(S,\eta) = \int_{K(S)\leq\eta} p_{K(S)}(y)dy \tag{3.1}$$

be the cumulative probability function of $K(S)$. Thus,

$$\eta_\beta(S) = \min\{\eta \in \mathbb{R} \,:\, \Psi(S,\eta) \geq \beta\} \tag{3.2}$$

and

$$\zeta_\beta(S) = (1-\beta)^{-1} \int_{K(S)\geq\eta_\beta(S)} K(S)\, p_{K(S)}(y)\, dy. \tag{3.3}$$

Let $S^* = \arg\min_S \zeta_\beta(S)$ and $\zeta^*_\beta = \min_S \zeta_\beta(S) = \zeta_\beta(S^*)$. The first research question can now be succinctly re-stated as what is the $S^*$ that leads to $\zeta^*_\beta$? In the lines below, we will put this research question into perspective via a brief literature review of related work.

In recent years, a growing number of studies have applied the CVaR criterion to analyze risk-related problems in operations and supply chain management. Some of these studies deal with sourcing strategies (Tomlin and Wang, 2005), channel coordination (Chen et al., 2014), and resource allocation (Wagner and Radovilsky, 2012). Other studies apply the CVaR criterion to inventory management settings, such as classical newsvendor models (Gotoh and Takano, 2007), newsvendor models with pricing decisions (Chen et al., 2009), newsvendor models with random capacity

(Wu et al., 2013), and multi-period inventory models (Borgonovo and Peccati, 2009; Zhang et al., 2009). However, none of these studies incorporate an important form of supply risk: an endogenous stochastic lead time, as the one found in the supply chain under study.

On the other hand, one can find many inventory and production papers dealing with random lead times and/or capacitated production systems. For example, Kaplan (1970) considers an inventory model with an exogenous stochastic lead time and characterizes the optimal policy. Many years later, Karmarkar (1987) explicitly models the supply process as a single-server queue, where the lead time is endogenous and dependent on the capacity of the production system and on the order size. Zipkin (1986) parallels Karmarkar's approach but in a more general production setting modeled as a queuing network. Lee and Zipkin (1992, 1995) examine serial queues and network of queues, and obtain tractable approximations of the system performance. Arreola-Risa (1996) analyzes a multi-period production-inventory model with multiple products and a capacitated production system. Nevertheless, all of these papers pursue minimization of expected cost, which is equivalent to assuming the decision maker is risk-neutral.

To summarize, to the best of our knowledge, our paper is the first one to combine a stochastic and capacitated supply chain with a risk-averse decision maker who follows the CVaR criterion.

## 3.3 Finding the optimal base-stock level and optimality of a stockless operation

Let $OO$ denote the number of outstanding production orders at the production facility. Because the demand and production rates are Poisson processes and each arriving demand triggers a production order for one unit, the steady-state distribution of $OO$ is equivalent to the steady-state distribution of the number of customers in a $M/M/1$ queuing system. Hence in steady state, $OO$ is geometrically distributed with parameter $\rho \equiv \lambda/\mu$ at any point in time. The parameter $\rho$ corresponds to the average utilization of the production process capacity, which for brevity will be called *capacity utilization*.

Let $I(S)$ be the instantaneous (inventory holding and demand back-ordering) cost rate given $S$, and $p_{I(S)}(\cdot)$ be the probability distribution of $I(S)$ at any point in time. In the next proposition, we establish an expression for $p_{I(S)}(\cdot)$.

*Proposition* 3.3.1. For the supply chain under study

$$p_{I(S)}(y) = \begin{cases} (1-\rho)\rho^x, & \text{if } y = h(S+1-x) \textit{ and } 0 \le x \le S \\ (1-\rho)\rho^x, & \text{if } y = b(x-S-1) \textit{ and } x \ge S+1 \end{cases} \tag{3.4}$$

At first sight, it appears that $I(S)$ is geometrically distributed. Unfortunately, further inspection of Proposition 3.3.1 indicates that $p_{I(S)}(\cdot)$ does not resemble any of the known probability mass functions. In addition, a moment's reflection reveals that to determine $p_{K(S)}(\cdot)$, one would need to take convolutions of $p_{I(S)}(\cdot)$, which is patently intractable.

To understand the intractability, let's define the instantaneous cost rate at time $t = \theta$ as $I(S,\theta) = I(S)|_{t=\theta}$ and let $T$ be equal to one year. Then $K(S) = \int_0^T I(S,\theta)d\theta = I(S,0)\cdot T + \int_0^T [I(S,\theta) - I(S,0)]d\theta$ for a time horizon $[0,T]$. Because $I(S,\theta)$ may have changed due to demand arrivals and production completions as $\theta$ goes from 0 to $T$, and it is well-known that the state-transition behavior of an $M/M/1$ queue is extremely complicated (Abate and Whitt, 1987, 1988; Leguesdron et al., 1993), the integral $\int_0^T [I(S,\theta) - I(S,0)]d\theta$ is intractable.

To deal with this conundrum, we will assume that $I(S,\theta) - I(S,0) \approx 0$, which is true when $T$ is small, since $I(S,\theta) \to I(S,0)$ as $T \to 0$. We will approximate

$K(S)$ by $\hat{K}(S) = I(S,0)T = I(S)$ since $T = 1$ year. Substituting $K(S)$ by $\hat{K}(S)$ in Equations 3.1, 3.2 and 3.3, leads to

$$\hat{\Psi}(S,\eta) = \int_{\hat{K}(S)\leq\eta} p_{\hat{K}(S)}(y)dy \tag{3.5}$$

$$\hat{\eta}_\beta(S) = \min\{\eta \in \mathbb{R} \,:\, \hat{\Psi}(S,\eta) \geq \beta\} \tag{3.6}$$

$$\hat{\zeta}_\beta(S) = (1-\beta)^{-1} \int_{\hat{K}(S)\geq\hat{\eta}_\beta(S)} \hat{K}(S)\, p_{\hat{K}(S)}(y)\, dy \tag{3.7}$$

where the "hat" is used to denote an approximation. The minimizer of $\hat{\zeta}_\beta(S)$ will be denoted by $\hat{S}^*$. Recall that $S^*$ is the minimizer of $\zeta_\beta(S)$. Later in this section, we will test the accuracy of using $\hat{S}^*$ to estimate $S^*$.

Much to our dismay, a direct minimization of $\hat{\zeta}_\beta(S)$ using Equations 3.4, 3.5, 3.6 and 3.7 is still intractable. Fortunately, the intractability goes away when we use the "shortcut function" $F_\beta(S,\eta)$ proposed by Rockafellar and Uryasev (2002), where

$$F_\beta(S,\eta) = \eta + (1-\beta)^{-1}\mathbb{E}([\hat{K}(S) - \eta]^+) \tag{3.8}$$

and $[x]^+ = \max\{0,x\}^+$. For brevity, set $\Omega(S,\eta) = \mathbb{E}([\hat{K}(S) - \eta]^+)$. According to Rockafellar and Uryasev (2002), if $\{\hat{S}^*, \hat{\eta}^*_\beta\} = \arg\min_{\{S,\eta\}} F_\beta(S,\eta)$, then $\hat{S}^* = \arg\min_S \hat{\zeta}_\beta(S)$ and $\hat{\eta}^*_\beta = \hat{\eta}_\beta(\hat{S}^*)$, and it follows that $\hat{S}^* = \arg\min_S \Omega(S)$. We will first deal in Section 3.3.1 with minimization of $\Omega(S,\eta)$ with respect to $S$ for any $\eta$. Thereupon in Section 3.3.2 we will focus on minimizing $F_\beta(\hat{S}^*,\eta)$ with respect to $\eta$.

### 3.3.1 *Minimizing $\Omega(S,\eta)$ with respect to $S$ for any $\eta$*

Let $k_0 = S + 1 - \lceil \eta/h \rceil$ and $k_1 = S + 1 + \lceil \eta/b \rceil$. It is easy to show that if $h(S+1) \geq \eta$, then

$$\Omega(S,\eta) = \sum_{k=0}^{k_0} [h(S+1-k) - \eta](1-\rho)\rho^k + \sum_{k=k_1}^{\infty} [b(k-S-1) - \eta](1-\rho)\rho^k. \quad (3.9)$$

Similarly, it is easy to show that if $h(S+1) < \eta$, then

$$\Omega(S,\eta) = \sum_{k=k_1}^{\infty} [b(k-S-1) - \eta](1-\rho)\rho^k. \quad (3.10)$$

In the next proposition, we minimize $\mathbb{E}[K(S,\eta)]$ with respect to $S$ using Equations 3.9 and 3.10. Keep in mind that $k_1 - k_0 = \lceil \eta/b \rceil + \lceil \eta/h \rceil$ is a constant.

*Proposition* 3.3.2. When $\eta \leq h(S+1)$ the value of $S$ which minimizes $\Omega(S,\eta)$ is given by $\max\{S_1^*(\eta), S_2^*(\eta)\}$, where

$$S_1^*(\eta) = \lceil \eta/h \rceil - 1, \quad (3.11)$$

and

$$S_2^*(\eta) = \left\lfloor \frac{\ln\left(\frac{h}{1-\rho}\right) - \ln\left\{-\left[h\lceil \eta/h \rceil - \eta - \frac{h}{1-\rho}\right] + \rho^{k_1-k_0-1}\left[b\lceil \eta/b \rceil - \eta + \frac{b\rho}{1-\rho}\right]\right\}}{\ln(\rho)} \right\rfloor + \lceil \eta/h \rceil - 1. \quad (3.12)$$

On the other hand, when $\eta > h(S+1)$ the value of $S$ which minimizes $\Omega(S,\eta)$ is given by $S_3^*(\eta)$, where

$$S_3^*(\eta) = \lceil \eta/h \rceil - 2. \quad (3.13)$$

Recall that $\hat{S}^*$ is the minimizer of $\Omega(S,\eta)$ for any given $\eta$. Building on Proposition 3.3.2, in the next theorem below we establish the value of $\hat{S}^*$.

*Theorem* 3.3.3. The value of $S$ which minimizes $\Omega(S,\eta)$ for any given $\eta$ is equal to $\hat{S}_2^*(\eta)$ in Proposition 3.3.2. In other words,

$$\hat{S}^* = \left\lfloor \frac{\ln\left(\frac{h}{1-\rho}\right) - \ln\left\{-\left[h\lceil \eta/h\rceil - \eta - \frac{h}{1-\rho}\right] + \rho^{k_1-k_0-1}\left[b\lceil \eta/b\rceil - \eta + \frac{b\rho}{1-\rho}\right]\right\}}{\ln(\rho)} \right\rfloor$$
$$+ \lceil \eta/h\rceil - 1.$$

### *3.3.2 Minimizing $F_\beta(\hat{S}^*,\eta)$ with respect to $\eta$*

Using $\hat{S}^*$ in Theorem 3.3.3, we now proceed to optimize $F_\beta(\hat{S}^*,\eta)$ with respect to $\eta$. First, we have $F_\beta(\hat{S}^*,\eta) = \eta + (1-\beta)^{-1}\Omega(\hat{S}^*,\eta)$. Let $\hat{\eta}^* = \arg\min_\eta F_\beta(\hat{S}^*,\eta)$ and

$$\begin{aligned}
\Omega(\hat{S}^*,\eta) &= [h(\hat{S}^*+1)-\eta] - \rho^{k_0^*+1}\left[h\lceil \eta/h\rceil - \eta - \frac{h}{1-\rho}\right] - \frac{h\rho}{1-\rho} \\
&\quad + \rho^{k_1^*}\left[b\lceil \eta/b\rceil - \eta + \frac{b\rho}{1-\rho}\right] \\
&= hk_0^* + h\lceil \eta/h\rceil - \eta - \rho^{k_0^*+1}\left[h\lceil \eta/h\rceil - \eta - \frac{h}{1-\rho}\right] - \frac{h\rho}{1-\rho} \\
&\quad + \rho^{k_1^*}\left[b\lceil \eta/b\rceil - \eta + \frac{b\rho}{1-\rho}\right]
\end{aligned}$$

where $k_0^* = \hat{S}^* + 1 - \lceil \eta/h\rceil$ and $k_1^* = \hat{S}^* + 1 + \lceil \eta/b\rceil$. Since

$$\frac{\partial F_\beta(\hat{S}^*,\eta)}{\partial \eta} = 1 + \frac{-1+\rho^{k_0^*+1}-\rho^{k_1^*}}{1-\beta} \tag{3.14}$$

it is easy to see that $\frac{\partial}{\partial\eta}F_\beta(\hat{S}^*,\eta)$ is not necessarily continuous everywhere with respect to $\eta$ due to the discreteness of $\hat{S}^*$, $k_0^*$, and $k_1^*$. Consequently, the first-order condition $\frac{\partial}{\partial\eta}F_\beta(\hat{S}^*,\hat{\eta}^*) = 0$ is not useful and we need to find an alternative method to minimize $F_\beta(\hat{S}^*, \eta)$ with respect to $\eta$. For that purpose, in the theorem below we demonstrate that, despite the discreteness of $\hat{S}^*$, $k_0^*$, and $k_1^*$, $F_\beta(\hat{S}^*, \eta)$ is continuous in $\eta$. Moreover, in said theorem we establish membership of $\hat{\eta}^*$ in two sets.

*Theorem* 3.3.4. $F_\beta(\hat{S}^*, \eta)$ is continuous with respect to $\eta$. In addition, the local minimums of $F_\beta(\hat{S}^*, \eta)$ may only be located at $\eta = vh$ and $\eta = wb$, where $v, w \in \mathbb{N}$.

Given the state of affairs, in our quest for managerial insights we will now make two mild assumptions: $\rho \leq \beta$ and the ratio $q = b/h$ is an integer. Because usually $\beta \geq 0.9$ (see Rockafellar and Uryasev (2000, 2002) who say the most common values of $\beta$ are 0.90, 0.95 and 0.99), the first assumption should be satisfied in most practical applications. Regarding the second assumption, we will show later in the paper that the solution for cases where $q < b/h < q+1$ can be easily found by using the solutions with $q$ and $q+1$.

Let $m = \lfloor \eta/b \rfloor$ and $n = \eta - mq$. In Lemma 3.3.5, we establish the monotonicity of a special function useful for proving Theorem 3.3.6.

*Lemma* 3.3.5. The function $y(\eta) = \rho^{mq+n+m+1}[q + (\rho-1)n + \rho]$ is monotonically decreasing in $\eta = (mq+n)h$.

We are now prepared to postulate one of the fundamental results of this paper.

*Theorem* 3.3.6. When $\rho \leq \beta$ and $q = b/h$ is an integer, $\hat{S}^*$ is given by

$$\hat{S}^* = \inf\left\{S = (mq+n) - 1 \ : \ \rho^{mq+n+m+1}[q + (\rho-1)n + \rho] \leq 1 - \beta\right\} \tag{3.15}$$

and as a result

$$\hat{\eta}_\beta^* = \inf\left\{\eta = (mq+n)h \ : \ \rho^{mq+n+m+1}[q + (\rho-1)n + \rho] \leq 1 - \beta\right\} \tag{3.16}$$

For a set of parameter ($\lambda$, $\mu$, $\beta$, $h$ and $b$) values, to find $\hat{S}^*$ and $\hat{\eta}_\beta^*$ using Theorem 3.3.6 requires integer programming. As an alternative, in the corollary below we provide a short-cut which does not require integer programming.

*Corollary* 3.3.7. If $y\left((m'q + n' - 1)h\right) \leq 1 - \beta$, then $\hat{S}^* = m'q + n' - 2$ and $\hat{\eta}^*_\beta = (m'q + n' - 1)h$ and otherwise $\hat{S}^* = m'q + n' - 1$ and $\hat{\eta}^* = (m'q + n')h$, where

$$m' = \left\lfloor \left\lceil \frac{\ln(q+\rho) - \ln(1-\beta)}{-\ln \rho} \right\rceil / (q+1) \right\rfloor$$
$$n' = \max\left\{0, \left\lceil \frac{\ln(q+\rho) - \ln(1-\beta)}{-\ln \rho} \right\rceil - m'(q+1) - 1\right\}$$

After a long journey, we have arrived at an easy-to-use approximation of $B^*$, namely $\hat{B}^* = \hat{S}^* + 1$, where $\hat{S}^*$ is obtained from Theorem 3.3.6 and Corollary 3.3.7. In Section 3.3.4, we will present a simulation experiment to study the accuracy of $\hat{B}^*$ in estimating $B^*$.

### *3.3.3 Optimality of stockless operation*

Intuitively speaking, the manager, being risk-averse, would be tempted to at least have some product units in inventory to smooth the random interaction of demand and production. So the second research question is posed again: under which conditions, if any, would a stockless operation be optimal in the supply chain under study? The next proposition provides the answer.

*Proposition* 3.3.8. When $\rho \leq \beta$ and $q = b/h$ is an integer, a stockless operation is optimal if $\rho(q + \rho) < 1 - \beta$.

The condition in Proposition 3.3.8 indicates that it is indeed possible for a risk-averse manager to optimally run a stockless operation. For example, when $\beta = 0.9$ and $q = 1$, a stockless operation is optimal if $\rho = 0.05$. This insight means that if the manager had idle capacity at more than 95%, the buffering provided by such idle capacity to cope with financial risk would be enough, and thus no inventory would be required to provide any extra buffering. As a second example, when $\rho = 0.8$, $q = 1$, and $\beta = 0.9$, Proposition 3.3.8 says that the buffering provided by idle capacity at 20% is not enough to cope with financial risk (we know from the previous example that more than 95% is needed) and additional buffering would have to come from inventory, which essentially means a stockless operation is not optimal.

Note that the ratio $q = b/h$ measures the relative economic impact of a back-ordered unit when compared to the cost of having one unit in inventory. We will refer to $q$ as the *back-orders economic impact*. The condition in Proposition 3.3.8 also indicates that when a stockless operation is optimal, and hence all of the buffering against financial risk will come from idle capacity, the amount of idle capacity needed is increasing in the manager's risk sensitivity, and is also increasing in the back-orders economic impact. The logical implication is that if the manager's risk sensitivity was high, the capacity utilization was high, and the back-orders economic impact was high, a stockless operation being optimal would be extremely unlikely. This implication is intuitively pleasing and complements similar results reported in prior literature (Arreola-Risa and DeCroix, 1998; Rajagopalan, 2002; Arreola-Risa and Keblis, 2013).

### *3.3.4 Simulation experiment*

Because $\hat{B}^*$ is an approximation of $B^*$, we know that $\zeta_\beta(\hat{B}^*) \geq \zeta_\beta(B^*)$. We then would like to determine the penalty for using $\hat{B}^*$ instead of $B^*$, which will be measured as a percentage and calculated as

$$\%\ Penalty = \frac{\zeta_\beta(\hat{B}^*) - \zeta_\beta(B^*)}{\zeta_\beta(\hat{B}^*)}.$$

For this purpose, we conducted a simulation experiment on 54 supply chain scenarios which should be representative of most practical situations. The 54 supply chain scenarios resulted from all combinations of the following parameter values: $\rho$=0.5 and 0.9; $\mu$ =10, 100, and 1000; $h$=1; $b/h$=1, 5, 25; $\beta$=0.9, 0.95, and 0.99.

For each scenario, we first simulated 10,000 instances and collected the annual cost observed in each year (after discarding a warm-up period). Second, we constructed the probability distribution of annual cost from the 10,000 simulated annual costs. Next, we used the simulated annual cost probability distribution to find $B^*$ and to compute $\zeta_\beta(B^*)$ and $\zeta_\beta(\hat{B}^*)$. Lastly, we computed the % Penalty, where

$$\%Penalty = \frac{\zeta_\beta(\hat{B}^*) - \zeta_\beta(B^*)}{\zeta_\beta(B^*)} \tag{3.17}$$

The results of the simulation experiment are summarized in Tables 3.1 and 3.2, where the percentage of penalty is in parenthesis. The average penalty in Tables 3.1 is 1.58% and in Tables 3.2 is 0.43% for a grand average of 1.01%. These results suggest that $\hat{B}^*$ yields optimal or near-optimal solutions in a variety of supply chain scenarios and its accuracy should be acceptable in practical applications.

Table 3.1: Accuracy of $\hat{B}^*$ ($\rho = 0.5$)

| $\beta$ | $b/h$ | $\hat{B}^*$ | $B^*_{\mu=10}$ | $B^*_{\mu=100}$ | $B^*_{\mu=1000}$ |
|---|---|---|---|---|---|
| | 1 | 1 | 1 (0%) | 1 (0%) | 1(0%) |
| $\beta = 0.9$ | 5 | 4 | 4 (0%) | 4 (0%) | 4(0%) |
| | 25 | 7 | 7 (0%) | 6 (3.10%) | 6(1.96%) |
| | 1 | 2 | 2 (0%) | 2 (0%) | 2(0%) |
| $\beta = 0.95$ | 5 | 5 | 4 (0.36%) | 4 (3.93%) | 4(4.47%) |
| | 25 | 8 | 8 (0%) | 7 (5.47%) | 7(2.35%) |
| | 1 | 3 | 3 (0%) | 3 (0%) | 3(0%) |
| $\beta = 0.99$ | 5 | 7 | 7 (0%) | 6 (8.56%) | 6(4.97%) |
| | 25 | 10 | 10 (0%) | 9 (6.46%) | 9(0.94%) |

Table 3.2: Accuracy of $\hat{B}^*$ ($\rho = 0.9$)

| $\beta$ | $b/h$ | $\hat{B}^*_a$ | $B^*_{\mu=10}$ | $B^*_{\mu=100}$ | $B^*_{\mu=1000}$ |
|---|---|---|---|---|---|
| | 1 | 13 | 14 (0.65%) | 14 (0.56%) | 14(0.03%) |
| $\beta = 0.9$ | 5 | 32 | 32 (0%) | 31 (0.09%) | 30(0.84%) |
| | 25 | 50 | 50 (0%) | 50 (0%) | 48(0.39%) |
| | 1 | 17 | 17 (0%) | 17 (0%) | 16(0.62%) |
| $\beta = 0.95$ | 5 | 37 | 38 (0.02%) | 36 (0.07%) | 34(1.29%) |
| | 25 | 57 | 57 (0%) | 57 (0%) | 54(0.68%) |
| | 1 | 24 | 25 (0.87%) | 25 (0.51%) | 22(2.13%) |
| $\beta = 0.99$ | 5 | 50 | 51 (0.10%) | 50 (0%) | 45(2.75%) |
| | 25 | 71 | 70 (0.07%) | 70 (0.05%) | 70(0.01%) |

## 3.4 Post-solution analysis

Understanding the manager's optimal behavior regarding the decision variable $B^*$ when the supply chain setting changes is as important as finding the value of $B^*$. With that goal in mind, in this section we address the second research question: what should be the manager's optimal adjustment (if any) to $B^*$ as the values of the parameters $\lambda$, $\mu$, $\beta$, $h$ and $b$ change? We will use $\hat{B}^*$ to study the behavior of $B^*$. The first result is presented in Proposition 3.4.1.

*Proposition* 3.4.1. $\hat{B}^*$ is non-decreasing in $\beta$.

Everything else being equal, Proposition 3.4.1 states that if the manager's risk sensitivity increased, then his/her optimal strategy is to increase the optimal base-stock level. The quantitative explanation is that even though the annual cost probability distribution did not change, the manager now wants to minimize the average cost in a higher percentile, and doing so requires a higher optimal base-stock level. The qualitative explanation is that, even though the supply chain setting did not change (all other parameters stay the same), there is a psychological incentive to desire a higher optimal base-stock level because the manager's fear of financial risk got higher. The second result in presented in Proposition 3.4.2.

*Proposition* 3.4.2. $\hat{B}^*$ is non-decreasing in $q$.

Everything else being equal, Proposition 3.4.2 states that if the back-orders economic impact increases, then the manager's optimal strategy is to increase the optimal base-stock level. The quantitative explanation is that when the back-orders economic impact increases, the right tail of the annual cost probability distribution will increase as well, which naturally leads to a higher optimal base-stock level. The qualitative explanation is that when the back-orders economic impact increases, the inventory holding cost per unit now appears cheaper when compared to the back-ordering cost per unit, and consequently there is an economic incentive to increase the optimal base-stock level. The third result is presented in Proposition 3.4.3.

*Proposition* 3.4.3. $\hat{B}^*$ is non-decreasing in $\rho$.

Everything else being equal, Proposition 3.4.3 states that if capacity utilization increases, then the manager's optimal strategy is to increase the optimal base-stock level. The quantitative explanation is that when capacity utilization increases, the probability of incurring a large number of back-orders will increase, and with that, the right tail of the annual cost probability distribution will increase as well, which naturally leads to a higher optimal base-stock level. The qualitative explanation is that when capacity utilization increases, congestion in the production system will increase; with that, the average production lead time will increase as well, which leads to the need for a higher base-stock level.

## 3.5 Expected cost minimization and its consequences

As mentioned at the beginning of this paper, the use of expected cost in supply-chain management research is pervasive. Recall that $B^*_{EC}$ denotes the base-stock value which minimizes *expected* annual cost. So it is possible, say due to mathematical convenience and ease of calculation, that the manager in the supply chain under study would be tempted to use $B^*_{EC}$ instead of $B^*$. In this section, we examine the fourth and last research question: how does $B^*_{EC}$ compare to $B^*$ and how does $\zeta_\beta(B^*_{EC})$ compare to $\zeta_\beta(B^*)$? The answer to the first part of the fourth research question is provided in the proposition below using the approximation $\hat{B}^*$.

*Proposition* 3.5.1. $\hat{B}^* \geq B^*_{EC}$.

Although we define CVaR assuming $0 < \beta < 1$, in the proof of Proposition 3.5.1 we show that $\hat{B}^*$ converges to $B^*_{EC}$ when $\beta = 0$. When one combines the result in Proposition 3.5.1 with the result in Proposition 3.4.1, the following finding emerges: if the manager in the supply chain under study decided to use $B^*_{EC}$ instead of $\hat{B}^*$, he/she would be accepting a greater than optimal financial risk since the buffer provided by $B^*_{EC}$ would be less than the optimal or near-optimal buffer provided by $\hat{B}^*$. In addition, the higher the manager's risk sensitivity, the greater the difference between $\hat{B}^*$ and $B^*_{EC}$, and consequently the greater the unnecessary exposure to financial risk. This finding is illustrated in Figures 3.1 and 3.2. The step-wise pattern is due to $\hat{B}^*$ and $B^*_{EC}$ being integers.

The answer to the second part of the fourth research question eludes analytical treatment. However, Propositions 3.4.1 and 3.5.1 can be used again to arrive at the following finding: because $\hat{B}^* \geq B^*_{EC}$ and $\hat{B}^*$ is non-decreasing in $\beta$, we know that $\hat{\zeta}_\beta(B^*_{EC}) \geq \hat{\zeta}_\beta(\hat{B}^*)$ and $\hat{\zeta}_\beta(B^*_{EC}) - \hat{\zeta}_\beta(\hat{B}^*)$ is non-decreasing in $\beta$. In other words, the higher the manager's risk sensitivity, the higher the financial penalty he/she would pay for using $B^*_{EC}$ instead of $\hat{B}^*$. The finding is illustrated in Figures 3.3 and 3.4. The peculiar behavior of $\hat{\zeta}_\beta(B^*_{EC})$ in all three $\beta$ values is due to the fact that $B^*_{EC}$ is not optimal for minimizing CVaR.

Before closing this section, we want the reader to note that Figures 3.1 to 3.4 are

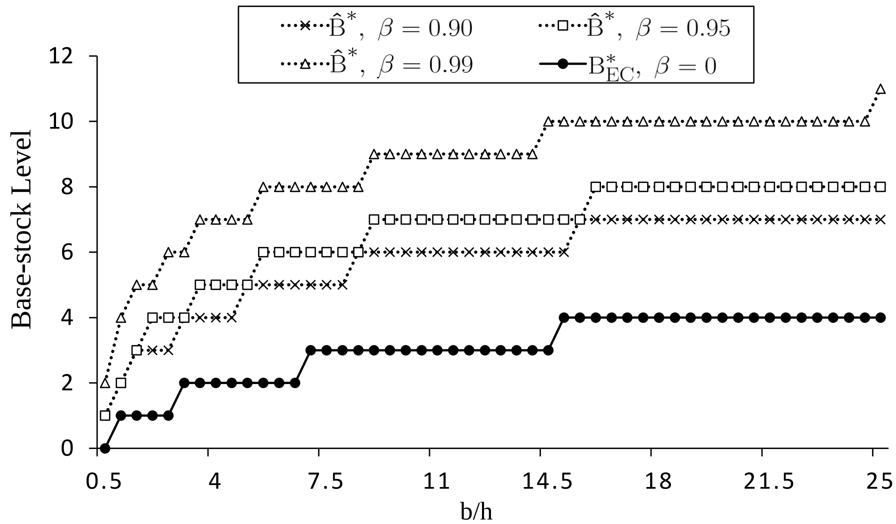



Figure 3.1: $\hat{B}^*$ vs. $B^*_{EC}$ ($\mu$=1, $\rho = 0.5$)

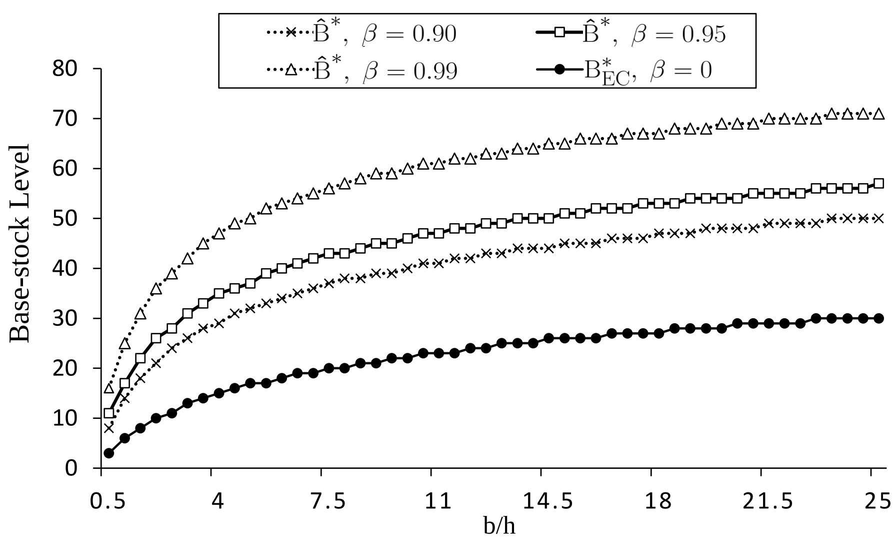



Figure 3.2: $\hat{B}^*$ vs. $B^*_{EC}$ ($\mu$=1, $\rho = 0.9$)

created by considering $b/h$ values from 0.5 to 25 in increments of 0.5. When $q = b/h$ is an integer, we use Corollary 3.3.7 to calculate $\hat{B}^*$. When $q = b/h$ is not an integer,

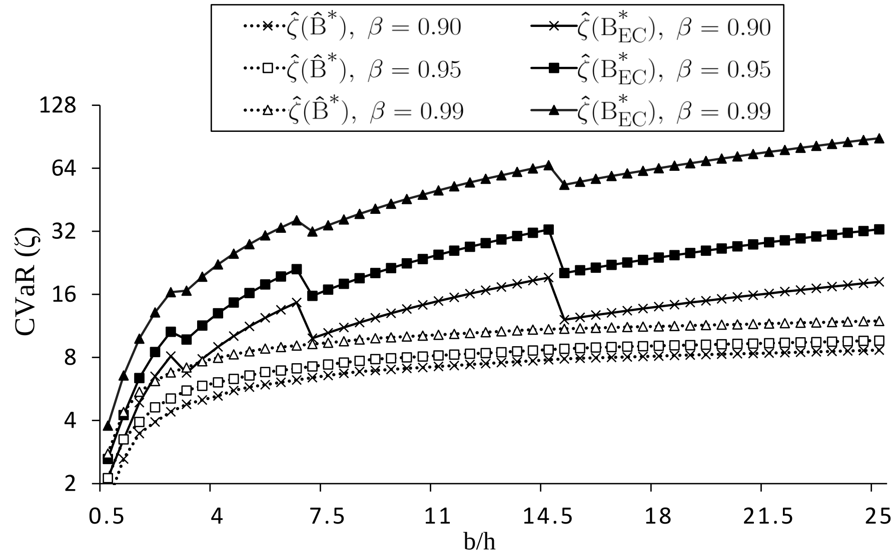



Figure 3.3: $\hat{\zeta}_\beta(\hat{B}^*)$ vs. $\hat{\zeta}_\beta(B^*_{EC})$ ($\mu$=1, $\rho = 0.5$)

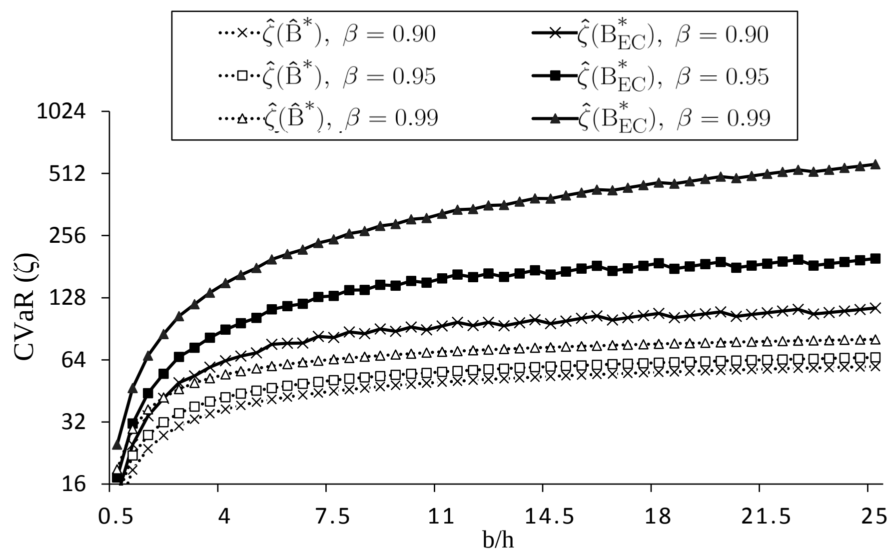



Figure 3.4: $\hat{\zeta}_\beta(\hat{B}^*)$ vs. $\hat{\zeta}_\beta B^*_{EC})$ ($\mu$=1, $\rho = 0.9$)

we use Theorem 3.3.4 to calculate $\hat{B}^*$ numerically. As mentioned in the paragraph after Theorem 3.3.4, we were able to easily obtain the optimal base-stock levels by using the solutions of problems with $q_1 = \lfloor b/h \rfloor$ and $q_2 = \lceil b/h \rceil$. We observe that $\hat{B}^*|_{q=q_1} \leq \hat{B}^*|_{q=b/h} \leq \hat{B}^*|_{q=q_2}$ as shown in Figures 3.1 and 3.2.

## 3.6 Summary and conclusions

We considered a stochastic and capacitated supply chain whose manager is risk-averse and wants to find $B^*$: the base-stock level which minimizes the CVaR of annual cost. We showed that finding $B^*$ is an intractable problem and we developed an easy-to-use approximation of $B^*$ denoted by $\hat{B}^*$. We conducted a simulation experiment in which $\hat{B}^*$ yielded optimal or near-optimal solutions in a variety of scenarios of the supply chain under study. Given the aforementioned accuracy of $\hat{B}^*$, we used $\hat{B}^*$ to study the optimality of a stockless operation, to gain an understanding of the manager's optimal strategies when the supply chain parameters change, and to explore the consequences for the manager of minimizing expected annual cost instead of minimizing the CVaR of annual cost.

The research findings and managerial insights are many and diverse. We derived a simple condition for the optimality of a stockless operation and showed that it is possible to both eliminate inventory and achieve lowest CVaR for managers concerning the working capital tied up in inventories. By analyzing the optimal solution, we learned that the manager's optimal strategy is to increase the buffering provided by inventory when the manager's risk sensitivity increases, the back-orders economic impact increases, or the capacity utilization increases. Therefore, managers need to monitor the supply chain's operating characteristics and respond accordingly when they change. We also found that if the risk-averse manager were to minimize expected annual cost instead of minimizing the CVaR of annual cost, he/she would end up with a greater-than-optimal financial risk, which is not desirable and highlights the importance of using the CVaR-minimizing base-stock level suggested in this paper.

Because this paper is the first one to combine a stochastic and capacitated supply chain with a risk-averse manager who follows the CVaR criterion, the directions for further research are plentiful. We will list a few potential directions. One may consider other risk management tools such as the mean-variance trade-off or option pricing. The CVaR criterion could also be applied to study supply chain disruptions and resilience. Given the identified link between capacity utilization and CVaR of annual cost via the optimal base-stock level, the link could be utilized to investigate the capacity investment problem. Lastly, having focused in this paper on the cost over a short-term horizon, to consider long-term projects where the manager would be interested in minimizing the CVaR of the total cost over say twenty or thirty years may be a worthwhile research pursuit.

# 4. BONE MARROW TRANSPLANTATION WITH FINITE WAITING ROOMS

## 4.1 Introduction

As the eighth most common type of cancer in women and the tenth most common in men (Kasteng et al., 2007), leukemia is a very common cancer of the blood and bone marrow leading to build-up of abnormal white blood cells (National Cancer Institute, 2014). Siegel et al. (2015) estimate that a total of 54,270 leukemia cases occurred in the United States in the year 2015, of which 30,900 cases are male and 23,370 cases are female. It is projected that a total of 24,450 deaths due to leukemia will occur in the United States in the year 2015 (Siegel et al., 2015). At its current rate of incidence, it is estimated that approximately 1.5% of men and women will be diagnosed with leukemia at some point during their lifetime, with a five-year survival rate of 58.5% (National Cancer Institute, 2016). Older adults above age 65 constituted 52.9% of new leukemia cases during 2007-2011 (National Cancer Institute, 2016). Moreover, leukemia is among the most common childhood cancers and 10.1% of new leukemia cases are patients below age 20. Hence, leukemia is an important social issue and its cure is sought after by medical communities. Routine treatment methods of leukemia include monitored waiting, targeted therapy, radiation therapy and chemotherapy, all of which may block the disease progression but provide no cure. However, patients with leukemia may potentially be cured by stem cell transplant (National Cancer Institute, 2014). The operational challenge of a stem cell transplant lies in designing the health care delivery system to maximize patient health benefits. In this paper, we investigate how the design of medical units can impact the health benefits received by leukemia patients, which has implications for health care institutions, leukemia patients, and the society in general.

Hematopoietic Stem Cell Transplantation (HSCT), used to treat approximately 50,000 people worldwide in 2006 (Appelbaum, 2007), is an effective and popular method for treating leukemia and restoring the patient's ability to produce new healthy blood cells. HSCT consists of primarily bone marrow transplantation (BMT) and secondarily umbilical cord blood transplantation.

The process of HSCT begins with harvesting healthy stem cells from the patient him/herself or other individuals; with a sufficient amount of healthy stem cells, the next step is transplantation of the stem cells, which is followed by severe myelosuppression; the final step is engraftment, where the transplanted healthy stem cells thrive in bone marrows and replace the cancer cells. The aim of HSCT is the elimination of the underlying disease in the treatment recipient, together with full restoration of hematopoietic and immune function (Duncombe, 1997).

Medical centers specializing in cancer treatment often have a BMT unit with multiple rooms for surgery and recovery of patients, and they need to maintain a certain number of waiting rooms (wards under observation) for incoming patients to stay before receiving BMT. In the non-profit medical center that motivates this study, patients arrive randomly and receive BMT if there is a transplant room available at their time of arrival, or wait in one of the waiting rooms until a transplant room becomes available to serve him/her. A transplant room hosts both the surgery and recovery of a patient, since BMT requires a dedicated aseptic dust-free environment that only a specially-designed transplant room can provide (PWI Engineering, 1997; Dykewicz et al., 2000). The length of stay (LoS) of patients receiving BMT in transplant rooms is usually several weeks.

However, having patients experience long waits prior to a transplant is undesirable, since patients' medical conditions may deteriorate while waiting in a ward. Medical centers do not want to keep a high number of waiting rooms, which may lead to long waiting times. However, rejecting patients due to full waiting rooms is also costly for both the medical center and the patient rejected, since the patient may end up with limited and less beneficial alternatives.

The aim of this study is to investigate with queuing analysis how the performance of a BMT unit is impacted by the number of waiting rooms and the number of transplant rooms. We first formulate the problem with a constant number of transplant rooms. Our first research question is: *in the short run, how many waiting rooms should be allocated to maximize patient health benefits?* To answer this question, we optimize the number of waiting rooms from a societal perspective (Siegel et al., 1996) of maximizing the health benefits of the patients. Next, we explore the second research question: *in the long run, what is the trade-off between infrastructure investment and patient health benefits?* We answer this question by illustrating how the number of transplant rooms can impact the health benefits of patients with different arrival rates to account for the growth of the patient base. We verify the approximations we use via numerical simulation. We demonstrate that our analysis can be useful for improving the performance of health care organizations.

We contribute to the literature in several fronts. First, we analyze a queuing-based model with the objective of maximizing health benefits provided by the medical center. We then predict system performance and propose methods to optimize the number of waiting rooms. Second, we use sensitivity analysis to identify which parameters most significantly affect the optimal policy and prioritize improvement efforts. Third, we examine the option of adding/closing transplant rooms as patient arrival rates change and demonstrate the cost/benefit trade-off under this scenario. Lastly, we explore the sensitivity of system performance and the cost/benefit trade-off to system parameters and provide managerial recommendations.

The remainder of the essay is organized as follows. Section 4.2 reviews the relevant literature on health-care applications of queuing theory and relevant literature on approximation methods for queuing models with finite waiting rooms. Section 4.3 describes the model for optimizing the number of waiting rooms, and Section 4.4 describes the model for assessing the impact of the number of transplant rooms. In Section 4.5, we summarize key findings and discuss the limitations of this study.

## 4.2 Literature review and background

Operational analysis may add significant value to health care services (Green, 2012) by improving the performance (Porter, 2010) in areas such as medical decision-making (Zhang et al., 2012), surgical scheduling (Chow et al., 2011; Day et al., 2012), hospital admission control (Helm et al., 2011), nurse staffing (Wright et al., 2006), and congestion in patient transfer between different inpatient care units (Price et al., 2011; Bretthauer et al., 2011).

### *4.2.1 Queuing analysis in healthcare*

Among all approaches in operations management, queuing analysis has grown to be a popular method in modeling healthcare delivery systems, since it can be of great value in helping healthcare organizations to manage resource utilizations and patient flow delays (Green, 2006). Prior studies have shown the effectiveness of queuing analysis in the management of emergency departments (Wiler et al., 2011), outpatient clinics (Dobson et al., 2012), organ transplant (Su and Zenios, 2004), bed management (Cooper and Corcoran, 1974), and pharmacy (Shimshak et al., 1981). Significant interest lies in the performance evaluation and optimization of healthcare systems using queuing theory. Griffiths et al. (2006) model an Intensive Care Unit (ICU) as a $M/H/c/\infty$ queuing system and find that more nurses should be scheduled for each shift to avoid the costly ad-hoc need of supplementary nurses. Su and Zenios (2002) analyze a medical queuing system with patient reneging autonomously; they find that the Last-Come First-Served (LCFS) discipline may be the socially-optimal queuing discipline rather than the First-Come First-Served (FCFS) discipline. Dobson et al. (2012) consider the batching of patients in a medical teaching facility and show that in systems with limited buffer space, large batches can sometimes degrade efficiency by simultaneously increasing flow time and decreasing throughput, though throughput generally increases with batch size. Mandelbaum et al. (2012) consider a hospital with several heterogeneous wards that vary in service quality and speed, who use a quality- and efficiency-driven regime to describe the patient routing process and propose routing algorithms that take into account fairness towards hospital staffs. Véricourt and Jennings (2011) find that effective staffing policies should deviate from threshold-specific nurse-to-patient ratios, which are stipulated in some legislations, by taking into account the number of patients in the care unit. Gorunescu et al. (2002) model the geriatric department of a hospital with no waiting room and optimize the number of inpatient beds using the Erlang loss formula. They demonstrate that the bed-count decision is analogous to setting the base-stock level for an inventory system. None of these studies explore operational decisions such as the number of waiting rooms and transplant rooms. We also differ from extant literature in using quality-adjusted life years as the unit of measure instead of crude measures such as mortality rate, which enables the decision-maker to maximize the aggregate health benefit of all patients. For a comprehensive review of applications of queuing theory in health care, see Appa Iyer et al. (2013).

### *4.2.2 Approximation of finite-waiting-room queuing systems*

In this subsection, we outline the extant approaches for analyzing finite-waiting-room queuing systems and the approach we adopt in this paper. Queuing systems with finite waiting rooms have been examined by many studies despite technical difficulties associated with the truncation of waiting room size. As the simplest case, Morse (1958) develops some closed-form expressions for exponential systems and obtains results for optimizing the system, with waiting room size and service rate as decision variables. Köchel (2004) proves several monotonic properties for $M/M/c/K$ systems. Jouini et al. (2007) prove that for an $M/M/c/r+M$ system, the probability of being served is strictly increasing and concave in the waiting room size. For queuing systems with finite waiting rooms, researchers have made considerable efforts in obtaining useful approximations for $M/G/c/K$ and $G/G/c/K$ systems, since the performance measures are often intractable.

Smith (2004) discusses the optimal design and performance modeling of an $M/G/1/K$ queuing system. Closed-form expression of the performance measures of the $M/G/1/K$ system (such as blocking probability) cannot be obtained. Therefore, Smith develops a two-moment approximation based on approaches of Tijms (1992) and Kimura (1996). Similarly, an exact solution of the $M/G/c/K$ queuing system is only possible with exponential service or a single server, or no waiting room at all. Smith (2003) presents an approximation of the $M/G/c/K$ queuing system based on a closed-form expression of the finite capacity exponential queue. The approximation performs well with small number of servers, but becomes too complex for larger numbers of servers. Smith (2007) surveys the optimal design of the $M/G/c/K$ queuing system. Stidham (1992) considers the design of arrival rates and service rates, as an extension of Dewan and Mendelson (1990).

Choi et al. (2005) present a simple two-moment approximation for some important performance measures of a $GI/G/c/K$ queue, such as the loss probability, the mean queue length, and the mean waiting time. Choi et al. also show that the approximation is extremely simple yet satisfactory in its performance with extensive numerical studies. Sakasegawa et al. (1993) present an approximation formula for the blocking probability of $GI/GI/c/K$ queues. Whitt (2005) considers an $M/G/s/r+GI$ queue, which has a Poisson arrival process, i.i.d. service times with a general distribution, $s$ servers, $r$ extra waiting spaces, and *i.i.d.* customer abandonment times with a general distribution. Whitt shows that the $M/G/s/r+GI$ queue can be approximated by $M/M/s/r+M(n)$ queue with state-dependent abandonment rates. The approximate distribution of waiting times can be numerically computed and simulation experiments show that the approximation is quite accurate.

### *4.2.3 Using quality-adjusted life years as a decision criteria*

We use quality-adjusted life years (QALYs), a health outcome measurement unit that combines duration and quality of life(Zeckhauser and Shepard, 1976), as the unit of measure of a BMT unit's performance. We use QALYs since the QALY maximization criterion is justified in a multi-attribute utility theory framework (Pliskin et al., 1980) and we can evaluate the benefit of BMT and the impact of waiting for BMT in QALYs for each patient on average. The use of QALYs is routine in cost-utility analysis of medical interventions and policies (Allen et al., 1989; Spiegelhalter et al., 1992; Broome, 1993; Singer et al., 1995; Räsänen et al., 2006; Sassi, 2006) as recommended by the U.S. Panel on Cost-Effectiveness in Health and Medicine (Lipscomb et al., 1996) and used in health-care operations (Packer, 1968; Fanshel and Bush, 1970; Zenios, 2002; Young and McClean, 2008; Zhang et al., 2012). Gold et al. (2002) provide a comprehensive overview of QALYs.

There are four major types of leukemia: acute lymphocytic leukemia (ALL), acute myeloid leukemia (AML), chronic myeloid leukemia (CML), and chronic lymphocytic leukemia (CLL). Several studies have examined the effectiveness of BMT compared with alternatives. Barr et al. (1996) examine the effectiveness of allogeneic BMT in AML versus no-treatment and estimate that the average benefit is 0.73 QALY.

Barr et al. (1996) also estimate that for ALL patients, allogeneic BMT brings an average benefit of 0.12 QALY compared to no-treatment. Lee et al. (1998) find that for CML patients, unrelated donor BMT leads to an average benefit of 5.25 QALYs versus Interferon-$\alpha$, a popular chemotherapy method. Based on these studies and the composition of four types of leukemia patients, we estimate the average benefit of the BMT procedure to be 1.435 QALYs per patient. For a comprehensive review of economic evaluations of leukemia, see Kasteng et al. (2007).

Next, we estimate the risk of death in leukemia and convert it into QALYs. We begin by estimating the mortality rate of leukemia patients using the five-year survival rate of 58.5% (National Cancer Institute, 2016). Assume the average mortality rate is $\gamma$ per day, we have $(1-\gamma)^{5\times 365} = 58.5\%$ and $\gamma = 2.94 \times 10^{-4}$. Assuming a discount rate of 3% and a life expectancy of 30 years, a statistical healthy life is equivalent to $\sum_{n=0}^{29}(1+0.03)^{-n} = 20.2$ years of healthy life using the method of Hirth et al. (2000). Thus, staying alive for one more day is on average equivalent to losing $1 + 20.2 \times 365 \times \gamma = 3.17$ days for a healthy person with 30 years of life-expectancy. For leukemia patients, we make quality-of-life adjustments based on CML patients' quality of life coefficient (0.5), as reported by Tengs and Wallace (2000), leading to 1.585 quality-adjusted-life-days (QALD) per day for leukemia patients.

#### *4.2.4 Two-moment approximation of the finite waiting room queuing system*

It is common to assume that the arrival process of patients is Poisson, with support from the literature. Çinlar (1968) proves that the superposition of many point processes converge to a Poisson process as the number of point processes approaches infinity. Albin (1982) shows that the expected delay of a $\Sigma GI_i/M/1$ queue can be approximated by an $M/M/1$ queue. Two approximation methods are proposed by Albin (1986).

In this paper, we also consider the case that the arrival process is a point process with generic distribution of inter-arrival times. Thus, we consider a $GI/G/c/K$ system in this paper, with general and independent inter-arrival times and $i.i.d.$ general service times to account for situations where the patient arrival process is not Poisson, such as a deterministic arrival pattern where arrivals are equally spaced in time. We develop expressions for system performance metrics, and then minimize the expected cost per unit time by optimizing the size of waiting rooms. To approximate $GI/G/c/K$ queues, we use the method of Choi et al. (2005), which is shown to be superior to other approximation methods. We show that the objective function of the (approximated) expected cost per unit time has the same structure as that of the $M/M/c/K$ queue, and that Taylor approximations can be used to locate the optimal solution.

We denote the patient arrival rate as $\lambda$ and service rate as $\mu$. We denote $A$ as the inter-arrival time of patients and $S$ as the service time of each patient. We also have the following notations:

We use some results and notations of the two-moment approximation of Choi et al. (2005), which is exact for exponential arrival and service processes. Denote

$$a_n^D \approx a_R = \frac{E[A^2]}{2E[A]} = \frac{(1+c_A^2)a}{2},\ 0 \leq n \leq c+r-1 \tag{4.1}$$

$$(b_n^A)b_n^D \approx b_R = \frac{E[S^2]}{2E[S]} = \frac{(1+c_S^2)b}{2},\ 1 \leq n \leq c+r-1 \tag{4.2}$$

We know that $a_{c+r}^D = a$ and $b_{c+r}^D = b$ when the system is full.

The system state probabilities $\widetilde{P}_n^A$, $\widetilde{P}_n^D$, $\widetilde{P}_n$ and parameters $\widetilde{\mu}_i$, $\widetilde{\lambda}_i$, and $\widetilde{\gamma}_i$ are outlined on page 78 of Choi et al. (2005).

Denote $B = \sum_{m=0}^{c-1} \prod_{n=1}^{m} \frac{b-n(a-a_R)}{na_R}$ and $D = \prod_{n=1}^{c-1} \frac{b-n(a-a_R)}{na_R}$. The probability of

Table 4.1: Key Notations

| Notation | Explanation |
|---|---|
| $A_n^D$: | residual inter-arrival time at departure epochs when number of customers in the system is $n$. |
| $P_n^A$, $P_n^D$, $P_n$: | probability of having $n$ customers at arrival epoches / departure epoches / arbitrary time. |
| $A_n^D$: | residual inter-arrival time at departure epochs when number of customers in the system is $n$. |
| $P_n^A$, $P_n^D$, $P_n$: | probability of having $n$ customers at arrival epoches / departure epoches / arbitrary time. |
| $a$ | $= E[A] = 1/\lambda$ |
| $b$ | $= E[S] = 1/\mu$ |
| $a_n^D$ | $= E[A_n^D]$, $0 \leq n \leq c + r$ |
| $b_n^A$ | $= E[S_n^A]$, $0 \leq n \leq c + r$ |
| $b_n^D$ | $= E[S_n^D]$, $0 \leq n \leq c + r$ |
| $\rho$ | $= \frac{\lambda}{c\mu} = \frac{\lambda b}{c}$, the nominal capacity utilization. |

system being empty at arrivals $\widetilde{P}_0^A(c, r)$, the parameters $\phi$, the loss probability $\widetilde{P}_{c+r}^A$, the mean queue length $\widetilde{L}_q$, and the probability of system having full transplant rooms but no patients waiting $\widetilde{P}_c^A$ are outlined in Equations 5.a to 5.e in Choi et al. (2005). We have $\phi = 1$ if $\rho = 1$.

Note that $b - c(a - a_R) = 1/\mu - c\left(1 - \frac{1+c_A^2}{2}\right)\frac{1}{\lambda} = \frac{c}{\lambda}\left(\rho - \frac{1-c_A^2}{2}\right)$. It follows that $b - c(a - a_R)$ is guaranteed to be positive when $c_A \leq 1$, which applies to most commonly-observed arrival patterns. Assuming $b - c(a - a_R) > 0$, we have that $B > 0$ and $D > 0$. Also note that $ca_R + b_R - b = \frac{c(1+c_A^2)a}{2} + \frac{(c_s^2-1)b}{2} = \frac{c}{\lambda}\left(\frac{1+c_A^2}{2} + \frac{c_S^2-1}{2}\rho\right)$. It follows that $ca_R + b_R - b > 0$ when $c_S \geq 1$ or $\rho < \frac{1+c_A^2}{1-c_S^2}$. In this paper, we restrict our attention to the case when $B > 0$, $D > 0$, $b - c(a - a_R) > 0$, and $ca_R + b_R - b > 0$.

## 4.3 Optimizing the number of waiting rooms

As previously discussed, the system we analyze is a $G/G/c/K$ queuing system. The total patient health benefit that the BMT unit generates per unit time is

$$\Pi(r) = B\lambda - C_1 L_q - C_2 \lambda P^A_{c+r}$$

and we want to find $r^* = \arg\max_{r \in \mathbb{N}} \Pi(r)$.

### *4.3.1 Using the two-moment approximation*

Instead of maximizing $\Pi(r)$, which is intractable, we maximize $\widetilde{\Pi}(r)$, the approximated version of $\Pi(r)$ using the two-moment approximation described in Section 4.2.4, since Choi et al. (2005) is a good approximation of $G/G/c/K$ queuing systems. We use the finite difference method to obtain $\widetilde{r}^* = \arg\max_{r \in \mathbb{N}} \widetilde{\Pi}(r)$, acknowledging that $r$ is discrete. For a function $F(\cdot)$, denote the difference between adjacent function values as $\Delta F(r) = F(r+1) - F(r)$. Our objective is to find $\widetilde{r}^* = \inf\{r \in \mathbb{N} : \Delta\widetilde{\Pi}(r) < 0\}$ such that when $r > \widetilde{r}^*$, adding more waiting rooms will not help. We present some analytical results in Lemmas 4.3.1 and 4.3.2.

*Lemma* 4.3.1. We have that when $\rho = 1$:

(a) $\Delta\widetilde{P}^A_0(r) = -\frac{ca_R}{ca_R+b_R-b}\widetilde{P}^A_0(r)\widetilde{P}^A_0(r+1)D$

(b) $\Delta\widetilde{P}^A_{c+r} = -\frac{a_R}{a}\frac{b-c(a-a_R)}{ca_R+b_R-b}\widetilde{P}^A_0(r)\widetilde{P}^A_0(r+1)D^2$

(c) $\Delta\widetilde{L}_q = \widetilde{P}^A_0(r)\widetilde{P}^A_0(r+1)D^2 \cdot \frac{ca_R\cdot[b-c(a-a_R)]}{ca_R+b_R-b} \cdot \left\{\frac{1}{2(ca_R+b_R-b)}r^2 + \left[\frac{B}{ca_RD} + \frac{1}{ca_R+b_R-b}\cdot\left(\frac{b_R-c(a-a_R)}{ca}+\frac{1}{2}\right)\right]r + \left[\frac{B}{ca_RD} + \frac{b_R-c(a-a_R)}{ca(ca_R+b_R-b)}\right]\left(1-\lambda a_R + \frac{ca_R+b_R-b}{ca}\right)\right\}$

(d) $\Delta\widetilde{\Pi} = -\widetilde{P}^A_0(r)\widetilde{P}^A_0(r+1)D^2 \cdot \frac{ca_R\cdot[b-c(a-a_R)]}{ca_R+b_R-b} \cdot \left\{\frac{C_1}{2(ca_R+b_R-b)}r^2 + \left[\frac{B}{ca_RD} + \frac{1}{ca_R+b_R-b}\cdot\left(\frac{b_R-c(a-a_R)}{ca}+\frac{1}{2}\right)\right]C_1\cdot r + C_1\left[\frac{B}{ca_RD} + \frac{b_R-c(a-a_R)}{ca(ca_R+b_R-b)}\right]\left(1-\lambda a_R + \frac{ca_R+b_R-b}{ca}\right) - C_2\lambda\cdot\frac{1}{ca}\right\}$

(e) $\widetilde{r}^* = 0$ if $B^2 - 4AC \leq 0$ and $\widetilde{r}^* = \left\lceil \frac{-B+\sqrt{B^2-4AC}}{2A} \right\rceil$ otherwise, where $A = -\frac{1}{2(ca_R+b_R-b)}$, $B = -\left[\frac{B}{ca_R D} + \frac{1}{ca_R+b_R-b}\left(\frac{b_R-c(a-a_R)}{ca} + \frac{1}{2}\right)\right]$, and $C = \left[\frac{B}{ca_R D} + \frac{b_R-c(a-a_R)}{ca(ca_R+b_R-b)}\right]\left(1 - \lambda a_R + \frac{ca_R+b_R-b}{ca}\right) - \frac{C_2\lambda}{caC_1}$.

We continue to examine the system performance when $\rho \neq 1$.

*Lemma* 4.3.2. We have that when $\rho \neq 1$:

(a) $\Delta\widetilde{P}_0^A = \widetilde{P}_0^A(r+1) - \widetilde{P}_0^A(r) = \widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1) \cdot \frac{\rho}{1-\rho}(\lambda a_R - 1 + \rho)(\phi - 1)D\phi^r$

(b) $\Delta\widetilde{P}_{c+r}^A = \widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)D^2(\phi-1)\phi^r \cdot \frac{b-c(a-a_R)}{ca} \cdot \left[\frac{B}{D} + \frac{1}{1-\rho}(\lambda a_R - 1 + \rho)\right]$

(c) $\Delta\widetilde{L}_q(r) = \widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)D^2\phi^r[b - c(a-a_R)] \cdot \Big\{ \frac{\rho}{1-\rho}(\lambda a_R - 1 + \rho)\left[\frac{1}{1-\phi}\frac{1}{ca_R+b_R-b} - \frac{1}{ca}\right]\phi^{r+1}\left[\frac{\phi-1}{ca} + \frac{1}{ca_R+b_R-b}\right] \cdot \left[\frac{B}{D} + \frac{1}{1-\rho}(\lambda a_R - 1 + \rho)\right] \cdot r + \left[\frac{1-\lambda a_R}{ca_R+b_R-b} + \frac{\phi}{ca}\right] \cdot \left[\frac{B}{D} + \frac{1}{1-\rho}(\lambda a_R - 1 + \rho)\right] + \frac{\rho}{1-\rho}\frac{(\lambda a_R-1+\rho)}{ca_R+b_R-b} \cdot \left[-\frac{1}{1-\phi} + \lambda a_R\right]$

(d) $\Delta\widetilde{\Pi} = -\widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)D^2\phi^r[b - c(a-a_R)] \cdot \Big\{ \frac{\rho}{1-\rho}(\lambda a_R - 1 + \rho)\left[\frac{1}{1-\phi}\frac{1}{ca_R+b_R-b} - \frac{1}{ca}\right]C_1\phi^{r+1}\left[\frac{\phi-1}{ca} + \frac{1}{ca_R+b_R-b}\right] \cdot \left[\frac{B}{D} + \frac{1}{1-\rho}(\lambda a_R - 1 + \rho)\right]C_1 \cdot r + \left[\left(\frac{1-\lambda a_R}{ca_R+b_R-b} + \frac{\phi}{ca}\right)C_1 + \frac{\phi-1}{ca}\lambda C_2\right] \cdot \left[\frac{B}{D} + \frac{1}{1-\rho}(\lambda a_R - 1 + \rho)\right] + \frac{\rho}{1-\rho}\frac{(\lambda a_R-1+\rho)}{ca_R+b_R-b} \cdot \left[-\frac{1}{1-\phi} + \lambda a_R\right]C_1$

### *4.3.2 The Taylor approximation methods*

Due to the complexity of the expressions of $\Delta\widetilde{\Pi}$, we propose three approximation methods based on the Taylor approximation to quickly identify the optimal number of waiting rooms.

(A) $\phi^r \approx 1 + r(\phi-1) + \frac{r^2-r}{2}(\phi-1)^2 = \frac{(1-\phi)^2}{2}r^2 + \frac{-\phi^2+4\phi-3}{2}r + 1$, which is Taylor approximation about $\phi$ in the neighborhood of $\phi = 1$. We obtain $\widetilde{r}_A = \arg\max_{r\in\mathbb{N}} \widetilde{\Pi}_A(r)$ with $\widetilde{\Pi}_A(r) \approx \widetilde{\Pi}(r)$ based on approximation (A).

(B) $\phi^r \approx \phi^{r_0}\left[1 + \ln\phi(r - r_0) + \frac{(\ln\phi)^2}{2}(r - r_0)^2\right]$, which is Taylor approximation about $r$ in the neighborhood of $r = r_0$. We use the optimal number of waiting rooms

Table 4.2: Accuracy of Approximations of $\widetilde{r}^*$

| $\rho$ | $c$ | Poisson arrival | | | | Deterministic arrival | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $\widetilde{r}^*$ | $\widetilde{r}_A$ | $\widetilde{r}_B$ | $\widetilde{r}_C$ | $\widetilde{r}^*$ | $\widetilde{r}_A$ | $\widetilde{r}_B$ | $\widetilde{r}_C$ |
| 0.6 | 10 | 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 0.6 | 60 | 33 | 33 | 33 | 33 | 33 | 33 | 33 | 33 |
| 0.8 | 10 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| 0.8 | 60 | 17 | 16 | 17 | 17 | 16 | 16 | 16 | 16 |
| 1 | 10 | 2 | - | - | - | 1 | - | - | - |
| 1 | 60 | 6 | - | - | - | 1 | - | - | - |
| 1.2 | 10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1.2 | 60 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |

when $\rho = 1$ as $r_0$. We then obtain $\widetilde{r}_B = \arg\max_{r\in\mathbb{N}} \widetilde{\Pi}_B(r)$ with $\widetilde{\Pi}_B(r) \approx \widetilde{\Pi}(r)$ based on approximation (B).

(C) Same with (B) except that we use $r_0 = \widetilde{r}_A$. We obtain $\widetilde{r}_C = \arg\max_{r\in\mathbb{N}} \widetilde{\Pi}_C(r)$ with $\widetilde{\Pi}_C(r) \approx \widetilde{\Pi}(r)$ based on approximation (C).

The three Taylor approximation methods can analytically find the optimal number of waiting rooms. We evaluate their performance by comparing them with the numerical solution without Taylor approximation and the simulation results.

### *4.3.3 Numerical study evaluating the impact of parameter values*

In this subsection, we conduct numerical studies with appropriate parameter values to study the impact of system parameters on the optimal solution. We consider four cases of nominal utilization ($\rho = 0.6/0.8/1.0/1.2$) to account for the various demand scenarios. We consider two cases of capacity level ($c = 10$ and $c = 60$) to account for various sizes of the bone marrow transplant department. We also explore different patient arrival patterns by considering both Poisson patient arrival and deterministic patient arrival. We present the results under Poisson arrival and deterministic arrival in Table 1.

We find that the three Taylor approximations behave satisfactorily in finding $\widetilde{r}^*$, meaning that using either approximation can greatly reduce the amount of computation needed while maintaining highly accurate solutions. This approximation can be quite helpful for managers who need quick insights for their proposed designs.

We also find that higher nominal capacity utilization leads to a lower number of waiting rooms in general, but also amplifies the impact of waiting rooms. This finding is illustrated in Figure 4.1 and 4.2. In Figure 4.1 and 4.2, sub-figures (A) and (B) represent Poisson arrival and Deterministic arrival, respectively. The trend that higher nominal capacity utilization amplifies the impact of waiting rooms does not always hold under Deterministic arrival, mainly because $\widetilde{r}^* = 1$ remains the same when $\rho$ increases from 1 to 1.2.

Comparing different patient arrival patterns, we discover that the optimal number of waiting rooms under Deterministic patient arrival is equal to or slightly lower than that under Poisson patient arrival. This finding is qualitatively intuitive since Deterministic patient arrival minimizes the uncertainty in the arrival process, but the magnitude of the difference between the two arrival patterns is shown to be negligible, meaning that managers do not need to pay much attention to patient arrival patterns.

## 4.4 Optimizing the number of transplant rooms

In this section, we consider a scenario where the BMT unit may adjust its number of transplant rooms, in addition to adjusting the number of waiting rooms. The BMT unit wants to balance between the total health benefit received by the arrived patients and the investment/savings associated with adjusting the number of transplant rooms.

There are two options to adjust the number of transplant rooms:

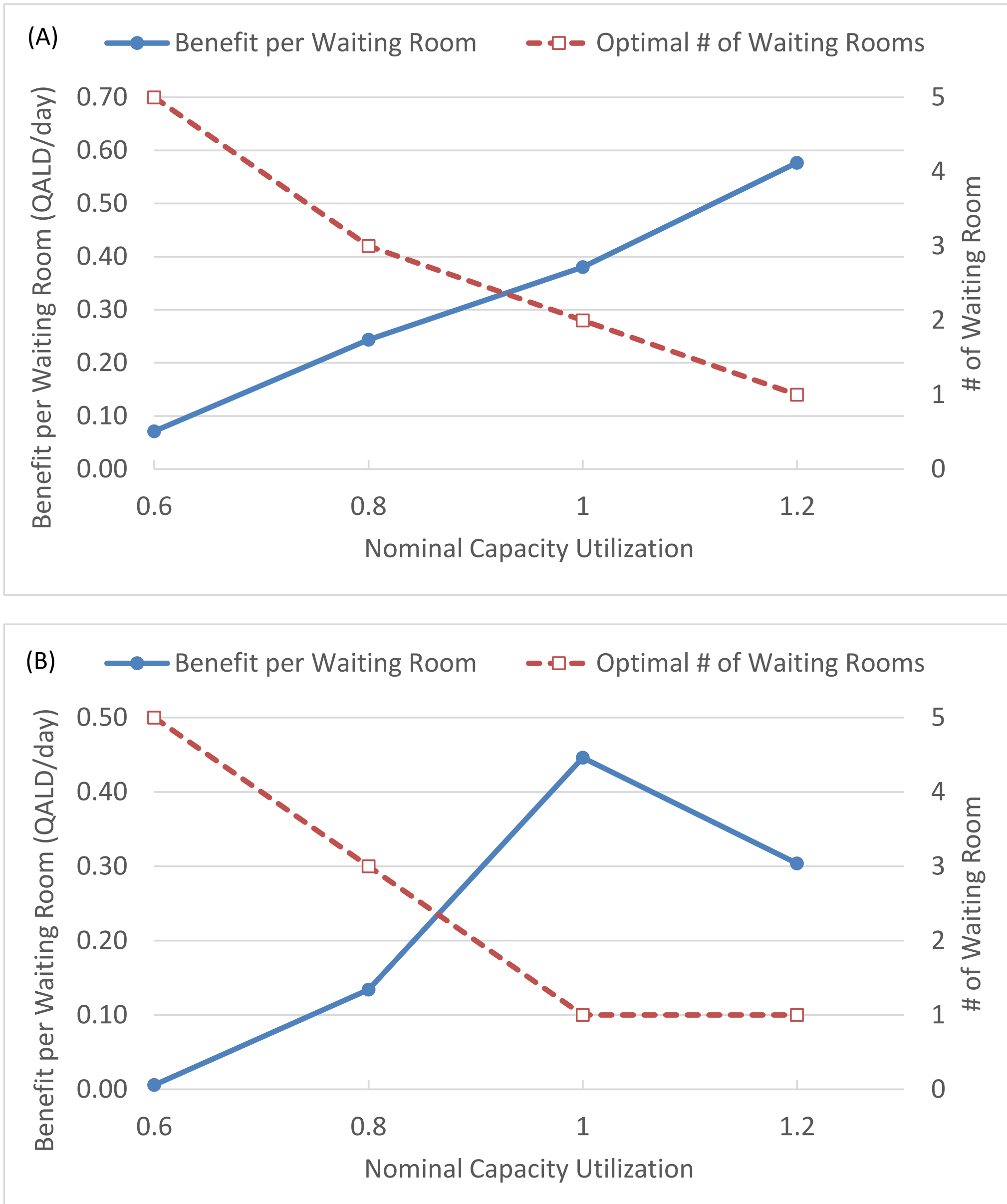



Figure 4.1: Impact of Utilization, Low-capacity Scenario (c=10)

1. Close some transplant rooms and yield cost savings;

2. Build additional transplant rooms at additional cost.

Note that building and closing transplant rooms take time and cannot be used as

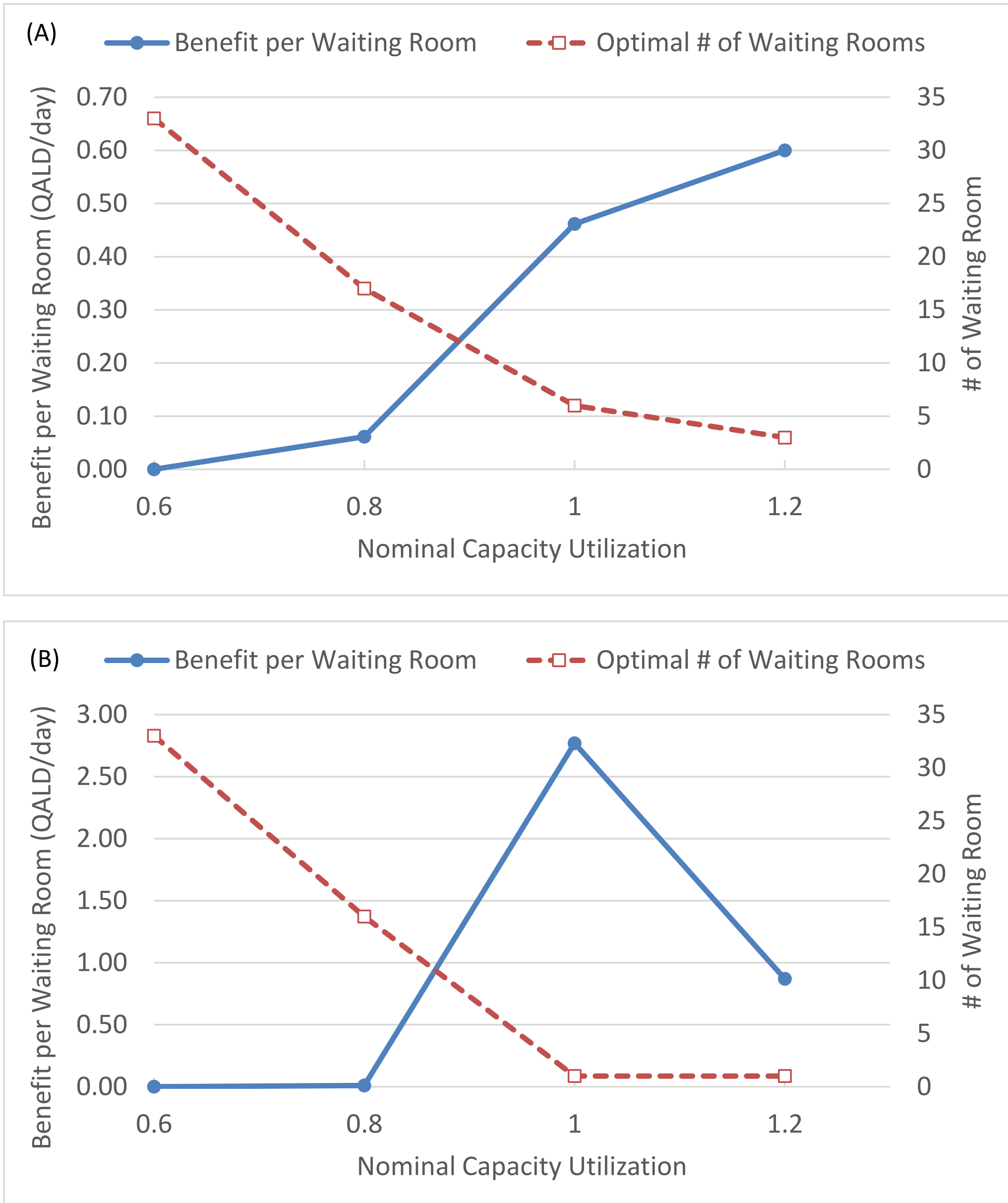



Figure 4.2: Impact of Utilization, High-capacity Scenario (c=60)

real-time adjustments of service capacity. Another question is, *when demand grows over time in the long-run, how will the optimal decision change?* Our model may also be used to optimize the design of new BMT units. Note that the expressions of

$\Delta\widetilde{\Pi}(r)$ are very complex except when $\rho = 1$. Even when $\rho = 1$, we can hardly use the expression to yield useful analytic results for adjusting the number of transplant rooms, since changing $c$ will result in a different value of $\rho$ and the expressions under $\rho = 1$ no longer apply.

We illustrate the trade-off between the two performance measures:

1. The health benefit created by the BMT unit: $\Pi(r, c) \approx \widetilde{\Pi}(r, c)$.

2. The financial impact involved in adding/closing transplant rooms:
   $B(c) = C_3(c - c_0)^+ + C_4(c_0 - c)^+$.

The cost of adding an additional transplant room and the cost of closing a transplant room are estimated using real data. The cost of adding/closing per square foot is \$250 $\sim$ \$300 (PWI Engineering, 1997), and we choose to use \$275 per sq ft as the estimate. The 10-bed unit at Sylvester Comprehensive Cancer Center in Miami, FL has an area of 9,600 square feet (Sylvester Comprehensive Cancer Center, 2016), from which we estimate the cost per bed as $C_3 = -C_4 = \$264,000$.

We present numerical results in Figures 4.3 to 4.10. We find diminishing returns in adding transplant rooms, especially when capacity utilization is low. We also notice that compared to Deterministic arrival, Poisson arrival is accompanied by a higher degree of diminishing returns in adding transplant rooms. For the number of waiting rooms, we discover that the optimal number of waiting rooms rises approximately linearly with the number of transplant rooms.

## 4.5 Summary and conclusions

We modeled a research center specializing in BMT as a $GI/G/c/K$ queuing system and analyzed its performance using a two-moment approximation method.

On the one hand, we maximized the health benefits of patients by choosing the

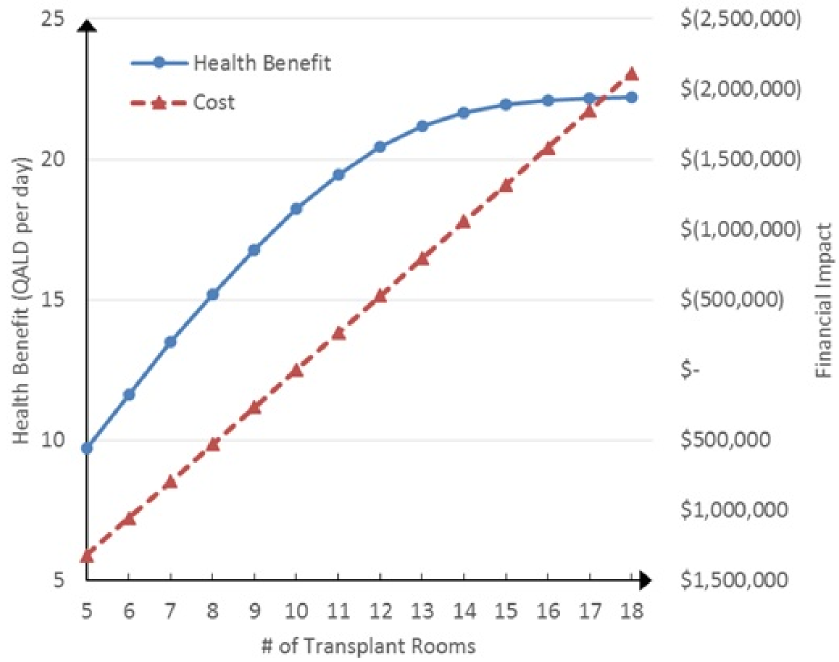



Figure 4.3: Small BMT Unit (c=10), Poisson Arrival, Cost

number of waiting rooms. We find that the number of waiting rooms impact the patients' health benefits in a non-linear manner. In particular, higher capacity utilization leads to a lower optimal number of waiting rooms, but amplifies each additional waiting room's marginal impact. We caution the managers that myopically adding more waiting rooms is not necessarily helpful, and can be very harmful especially when the capacity utilization is high.

On the other hand, we presented a cost-benefit trade-off for decision-makers when selecting different combinations of number of waiting rooms and number of transplant rooms. We find that the marginal benefits for adding more transplant rooms diminish quickly when capacity utilization is relatively low, which allows an administrator to rationalize the investment based on his/her trade-off between money and health benefits. Generally, a higher capacity utilization enables adding transplant rooms

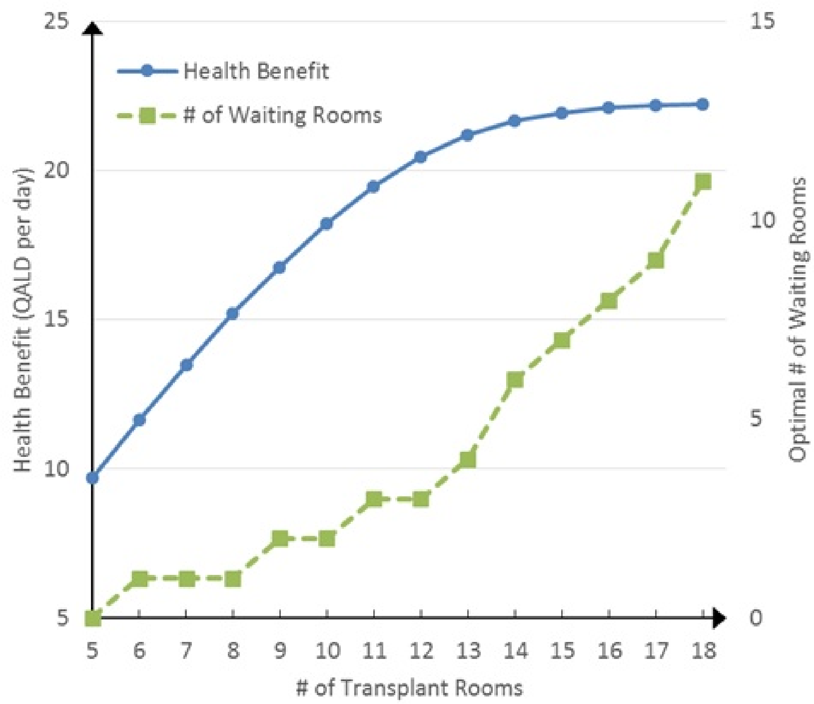



Figure 4.4: Small BMT Unit (c=10), Poisson Arrival, Waiting Room

to bring about more marginal benefits. An hospital administrator should evaluate the parameters accurately and avoid over-investment should the capacity utilization appears to be relatively low. We also find that more transplant rooms require more waiting rooms, indicating that these two types of resources are complementary, rather than substitutive.

Future research may examine how hospitals performing BMT can respond to cost-cutting efforts of health insurance providers. In particular, how current moves in medical reimbursement from fee-for-service to bundled-payment may urge hospitals design their health care delivery system more efficiently without sacrificing patients' health, where our techniques in this essay may prove helpful.

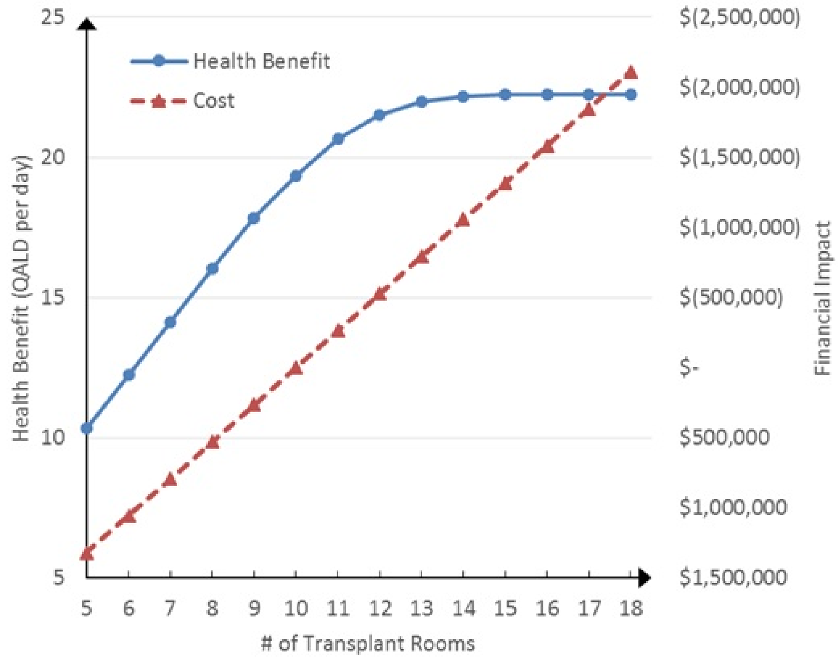



Figure 4.5: Small BMT Unit (c=10), Deterministic Arrival, Cost

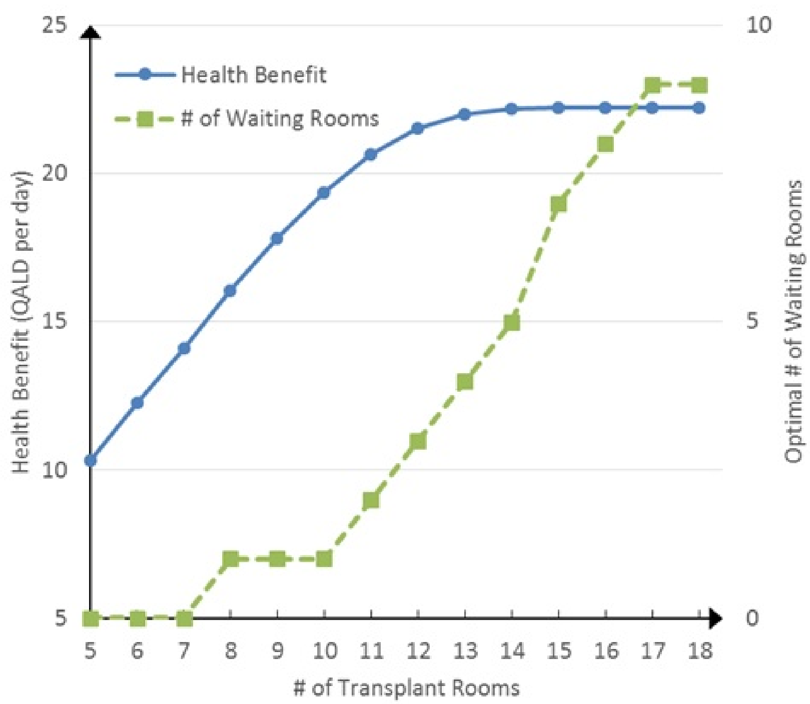



Figure 4.6: Small BMT Unit (c=10), Deterministic Arrival, Waiting Room

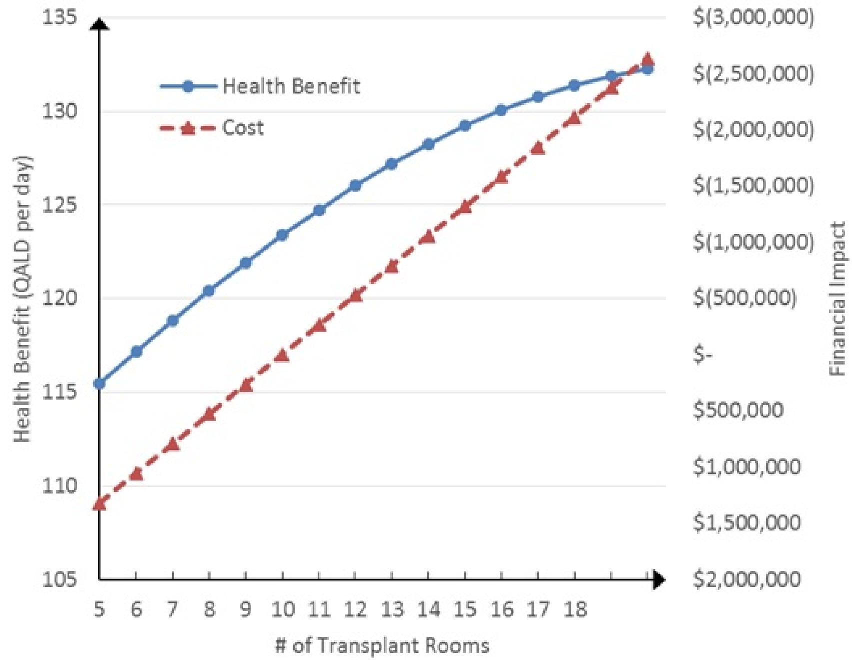



Figure 4.7: Large BMT Unit (c=60), Poisson Arrival, Cost

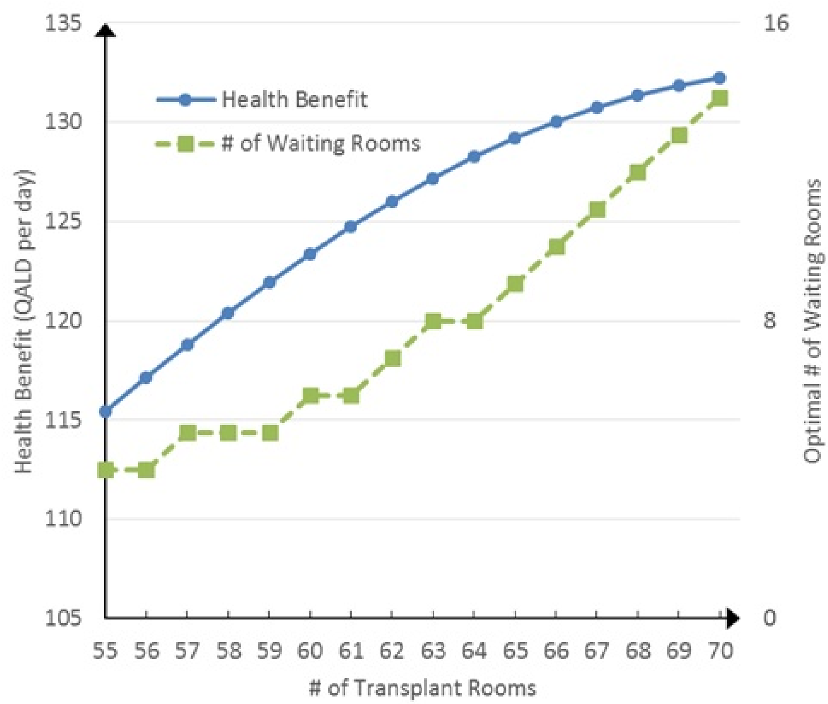



Figure 4.8: Large BMT Unit (c=60), Poisson Arrival, Waiting Room

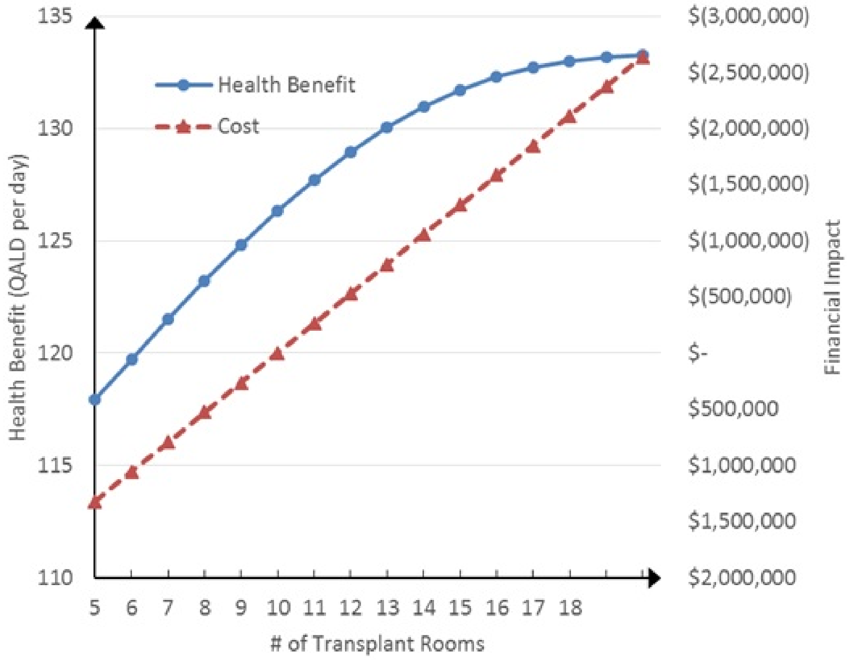



Figure 4.9: Large BMT Unit (c=60), Deterministic Arrival, Cost

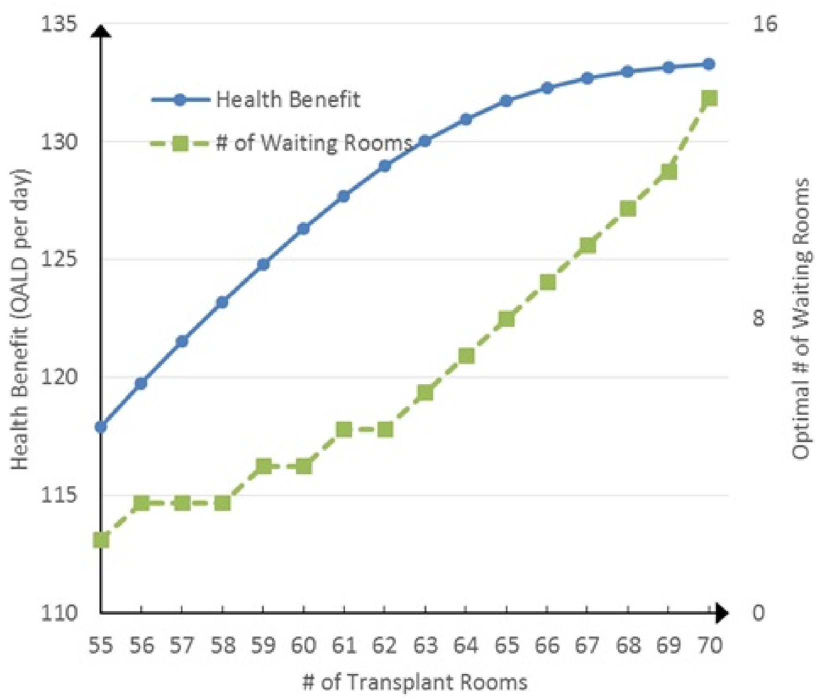



Figure 4.10: Large BMT Unit (c=60), Deterministic Arrival, Waiting Room

# 5. SUMMARY

I have included three essays in this dissertation. In my first essay, I analyze a firm facing inventory decisions under the influence of the financial market. With an analytical model, I examine the optimal inventory decisions under a variety of conditions and identify the relevant factors impacting such decisions and the firm's value. I also study the benefits brought by efforts to improve the random capacity of the firm. In my second essay, I model a stylized supply chain managed by a base-stock inventory policy where the decision makers concern about the down-side risk of the supply chain cost. I obtain solutions of the problem of minimizing Conditional Value-at-Risk under various supply chain scenarios. Moreover, I study the influence of supply chain parameters on the optimal solution as well as optimality of a stock-less operation. In my third essay, I investigate the operational decisions of a medical center specializing in bone marrow transplants. I formulate the medical center as a queuing system with random patient arrivals and departures. I then present optimal decisions and efficient frontiers regarding waiting room size and the number of transplant rooms with the objective of maximizing patient health benefits. In each of the three essays, I use analytical and numerical approaches to optimize managers' decisions with respect to various sources of risk.

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# APPENDIX A

## PROOFS OF SECTION 2

Proof of Lemma 2.3.1. Based on Equation 2.2, we have that:

$$(1 + r_f)S(Q) = [\mathbb{E}(D_1(Q)) - \Omega \text{Cov}(D_1(Q), M)] + [\mathbb{E}(V(Q)) - \Omega \text{Cov}(V(Q), M)] \tag{A.1}$$

$$\begin{aligned}(1 + r_f)\frac{\partial}{\partial Q}S(Q) &= \left[\frac{\partial}{\partial Q}\mathbb{E}(D_1(Q)) - \Omega\frac{\partial}{\partial Q}\text{Cov}(D_1(Q), M)\right] \\ &+ \left[\frac{\partial}{\partial Q}\mathbb{E}(V(Q)) - \Omega\frac{\partial}{\partial Q}\text{Cov}(V(Q), M)\right]\end{aligned} \tag{A.2}$$

**Part (I).** We first prove that $\frac{\partial}{\partial Q}\mathbb{E}[D(Q)] = [1-F(Q)][r-a(1+r_f)-(r-s)G(Q)]$. Denote the normalized random variables $y = [Y - \mu_Y]/\sigma_Y$ and $z = [Z - \mu_Z]/\sigma_Z$. We have that $\mathbb{E}(U(Q)) = [1-F(Q)]\cdot Q+\int_0^Q YdF(Y)$, $\mathbb{E}(D_1(Q)) = (1+r_f)[-a\cdot\mathbb{E}(U(Q))]$, and $\frac{\partial}{\partial Q}\mathbb{E}(U(Q)) = 1-F(Q)-f(Q)Q+Qf(Q) = 1-F(Q)$. Define auxiliary function $W(Q)$ such that $V(Q) = rU(Q) + (r - s)W(Q)$ where

$$W(Q) = \begin{cases} 0 & \text{if } Y \geq Q \text{ and } Z \geq Q \\ 0 & \text{if } Y < Q \text{ and } Z \geq Y \\ Z - Q & \text{if } Y \geq Q \text{ and } Z < Q \\ Z - Y & \text{if } Y < Q \text{ and } Z < Y \end{cases}$$

$$
\begin{aligned}
\mathbb{E}(W(Q)) &= \int_Q^{+\infty}\int_0^Q (Z-Q)f(Y,Z)dZdY + \int_0^Q\int_0^Y (Z-Y)f(Y,Z)dZdY \\
&= \int_Q^{+\infty}\int_0^Q (Z-Q)f(Z)dZf(Y)dY + \int_0^Q\int_0^Y (Z-Y)f(Z)dZf(Y)dY \\
&= [1-F(Q)]\left[\int_0^Q Zg(Z)dZ - \int_0^Q Qg(Z)dZ\right] \\
&+ \int_0^Q \left[\int_0^Y Zg(Z)dZ - \int_0^Y Yg(Z)dZ\right] f(Y)dY
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial Q}\mathbb{E}(W(Q)) = -f(Q)&\left[\int_0^Q Zg(Z)dZ - \int_0^Q Qg(Z)dZ\right] \\
&+ [1-F(Q)][Qg(Q) - Qg(Q) - G(Q)] \\
&+ f(Q)\left[\int_0^Q Zg(Z)dZ - \int_0^Q Qg(Z)dZ\right] \\
&= -[1-F(Q)]G(Q).
\end{aligned}
$$

Thus, we have that $\frac{\partial}{\partial Q}\mathbb{E}(D(Q)) = \frac{\partial}{\partial Q}\mathbb{E}(D_1(Q)) + \frac{\partial}{\partial Q}\mathbb{E}(V(Q)) = -a(1+r_f)\frac{\partial}{\partial Q}\mathbb{E}(U(Q)) + r\cdot\frac{\partial}{\partial Q}\mathbb{E}(U(Q)) + (r-s)\cdot\frac{\partial}{\partial Q}\mathbb{E}(W(Q)) = [1-F(Q)][r - a(1+r_f) - (r-s)G(Q)]$.

**Part (II).** Recall that $Y \sim F(Y) = \Phi\left(\frac{Y-\mu_Y}{\sigma_Y}\right)$, a normal distribution and $Z \sim G(Z) = \Phi\left(\frac{Z-\mu_Z}{\sigma_Z}\right)$, a normal distribution. Let $R(Q) = U(Q) - Q$, specifically $R(Q) = \min\{Y-Q, 0\}$. We know from Anvari (1987) that $\mathrm{Cov}(U(Q),M) = \mathrm{Cov}(R(Q)+Q,M) = \mathrm{Cov}(R(Q),M) = F(Q)\cdot\mathrm{Cov}(Y,M)$. It follows that $\mathrm{Cov}(D_1(Q),M) = -a(1+r_f)F(Q)\cdot\mathrm{Cov}(Y,M)$.

To maximize $S(Q)$, we need to find the expression $\partial\mathrm{Cov}(V(Q),M)/\partial Q$. Now we calculate $\mathrm{Cov}(V(Q),M)$. Since $\mathrm{Cov}(V(Q),M) = rF(Q)\mathrm{Cov}(Y,M) + (r-s)\mathrm{Cov}(W(Q),M)$, we focus on finding the expression of $\mathrm{Cov}(W(Q),M)$. We have that $\mathrm{Cov}(W(Q),M) = \mathbb{E}\{[W(Q) - \mathbb{E}(W(Q))][M - \mathbb{E}(M)]\} = \mathbb{E}\{W(Q)[M-\mathbb{E}(M)]\} - \mathbb{E}\{\mathbb{E}(W(Q))[M - \mathbb{E}(M)]\}$. Let $\mathbb{E}\{W(Q)[M-\mathbb{E}(M)]\} = \mathrm{Cov}_1(W(Q),M) + \mathrm{Cov}_2(W(Q),M) + \mathrm{Cov}_3(W(Q),M) +$

$\text{Cov}_4(W(Q), M)$, each part corresponding to one scenario of $W(Q)$. It is easy to show that $\mathbb{E}\{\mathbb{E}(W(Q))[M - \mathbb{E}(M)]\} = \mathbb{E}(W(Q)) \cdot \mathbb{E}\{[M - \mathbb{E}(M)]\} = 0$.

1) For the case of $Y \geq U$ and $Z \geq Q$, we have $\text{Cov}_1(W(Q), M) = \mathbb{E}\{[W(Q)][M - \mathbb{E}(M)]\}|_{Y \geq Q, Z \geq Q} = \mathbb{E}\{0 \cdot [M - \mathbb{E}(M)]\}|_{Y \geq Q, Z \geq Q} = 0$.

2) For the case of $Y < Q$ and $Z \geq Y$, denoting $y_Z = [Y - \mu_Z]/\sigma_Z$, we have $\text{Cov}_2(W(Q), M) = \mathbb{E}\{[W(Q)][M - \mathbb{E}(M)]\}|_{Y<Q, Z \geq Q} = \mathbb{E}\{0 \cdot [M - \mathbb{E}(M)]\}|_{Y<Q, Z \geq Q} = 0$.

3) For the case of $Y \geq Q$ and $Z < Q$, denote $y_Z = [Y - \mu_Z]/\sigma_Z$, we have $\text{Cov}_3(W(Q), M) = \mathbb{E}\{W(Q)[M - \mathbb{E}(M)]\}|_{Y \geq Q, Z<Q} = \int_0^Q \int_Q^{+\infty} \int_{-\infty}^{+\infty} [(Z - Q)]\sigma_M m \cdot f(m, Y, Z)\, dm\, dY\, dZ$.

Assume $M$, $Y$, and $Z$ are jointly normally distributed, $m$, $y$, and $z$ are the normalized standard normal variables. We have that:

$$
\begin{aligned}
\text{Cov}_3(W(Q), M) &= \sigma_M \int_{-\infty}^{q_Z} \int_{q_Y}^{+\infty} [\sigma_Z(z - q_Z)] \left[ \int_{-\infty}^{+\infty} m \cdot f(m|y, z)\, dm \right] f(y, z)\, dy\, dz \\
&= \sigma_M \int_{-\infty}^{q_Z} \int_{q_Y}^{+\infty} [\sigma_Z(z - q_Z)](y\delta_{MY} + z\delta_{MZ})\, \phi(y)\, dy\, \phi(z)\, dz \\
&= \sigma_M \int_{-\infty}^{q_Z} [\sigma_Z(z - q_Z)] \{\delta_{MY}\phi(q_Y) + z\delta_{MZ}[1 - \Phi(q_Y)]\}\, \phi(z)\, dz \\
&= \sigma_M \int_{-\infty}^{q_Z} [\sigma_Z \cdot z] z\delta_{MZ}[1 - \Phi(q_Y)]\, \phi(z)\, dz \\
&+ \sigma_M \int_{-\infty}^{q_Z} \{[\sigma_Z \cdot z\delta_{MY}\phi(q_Y) + [\sigma_Z(-q_Z)]z\delta_{MZ}[1 - \Phi(q_Y)]\}\, \phi(z)\, dz \\
&+ \sigma_M \int_{-\infty}^{q_Z} [\sigma_Z(-q_Z)][\delta_{MY}\phi(q_Y)]\, \phi(z)\, dz \\
&= \sigma_M\sigma_Z\delta_{MZ}[1 - \Phi(q_Y)][\Phi(q_Z) - q_Z\phi(q_Z)] \\
&+ \sigma_M \{[\sigma_Z\delta_{MY}\phi(q_Y) + [\sigma_Z(-q_Z)]\delta_{MZ}[1 - \Phi(q_Y)]\}\, [-\phi(q_Z)] \\
&+ \sigma_M[\sigma_Z(-q_Z)]\delta_{MY}\phi(q_Y)\Phi(q_Z) \\
&= \sigma_M\sigma_Z\delta_{MZ}[1 - \Phi(q_Y)]\Phi(q_Z) + \sigma_M\sigma_Z\delta_{MY}\phi(q_Y)[-\phi(q_Z) + \Phi(q_Z)(-q_Z)]
\end{aligned}
$$

4) For the case of $Y < Q$ and $Z < Y$, denote $y_Z = [Y - \mu_Z]/\sigma_Z$.

Case A: Assume $M$, $Y$, and $Z$ are jointly normally distributed, $m$, $y$, and $z$ are the normalized standard normal variables. We have that:

$$
\begin{aligned}
\mathrm{Cov}_4(W(Q), M) &= \mathbb{E}\{[W(Q)][M - \mathbb{E}(M)]\}|_{Y<Q,Z<Y} \\
&= \int_0^Q \int_0^Y \int_{-\infty}^{+\infty} [Z - Y]\sigma_M m \cdot f(m, Y, Z)\, dm\, dZ\, dY \\
&= \int_0^Q \int_0^Y [Z - Y] \left[\int_{-\infty}^{+\infty} \sigma_M m \cdot f(m|Y, Z)\, dm\right] f(Y, Z)\, dZ\, dY \\
&= \sigma_M \int_0^Q \int_0^Y [Z - Y] \left[\int_{-\infty}^{+\infty} m \cdot f(m|y, z)\, dm\right] f(y, z)\, dz\, dy \\
&= \sigma_M \int_{-\infty}^{q_Y} \int_{-\infty}^{y_Z} [[\sigma_Z \cdot z - \sigma_Y \cdot y + \mu_Z - \mu_Y]] \\
&\cdot \left[\int_{-\infty}^{+\infty} \sigma_M m \cdot f(m|y, z)\, dm\right] f(y, z)\, dz\, dy \\
&= \sigma_M \int_{-\infty}^{q_Y} \int_{-\infty}^{y_Z} \{[\sigma_Z \cdot z - \sigma_Y \cdot y + \mu_Z - \mu_Y]\} \\
&\cdot (y\delta_{MY} + z\delta_{MZ})\, \phi(z)\, dz\, \phi(y)\, dy \\
&= \sigma_M \int_{-\infty}^{q_Y} \int_{-\infty}^{y_Z} \sigma_Z \cdot z \cdot z\delta_{MZ}\, \phi(z)\, dz\, \phi(y)\, dy \\
&+ \sigma_M \int_{-\infty}^{q_Y} \int_{-\infty}^{y_Z} \{\sigma_Z \cdot z \cdot y\delta_{MY} - \sigma_Y \cdot y \cdot z\delta_{MZ}\}\, \phi(z)\, dz\, \phi(y)\, dy \\
&+ \sigma_M \int_{-\infty}^{q_Y} \int_{-\infty}^{y_Z} \{\{[\mu_Z - \mu_Y]\} \cdot z\delta_{MZ}\}\, \phi(z)\, dz\, \phi(y)\, dy \\
&+ \sigma_M \int_{-\infty}^{q_Y} \int_{-\infty}^{y_Z} \{[-\sigma_Y \cdot y + \mu_Z - \mu_Y]\} \cdot y\delta_{MY}\, \phi(z)\, dz\, \phi(y)\, dy
\end{aligned}
$$

We continue to further simplify the expression of $\text{Cov}_4(W(Q), M)$.

$$\begin{aligned}
\text{Cov}_4(W(Q), M) &= \sigma_M \sigma_Z \delta_{MZ} \int_{-\infty}^{q_Y} [\Phi(y_Z) - y_Z \phi(y_Z)]\, \phi(y)\, dy \\
&\quad + (\sigma_Y \delta_{MZ} - \sigma_Z \delta_{MY}) \sigma_M \int_{-\infty}^{q_Y} y \phi(y_Z)\, \phi(y)\, dy \\
&\quad + [\mu_Z - \mu_Y] \sigma_M \delta_{MZ} \int_{-\infty}^{q_Y} [-\phi(y_Z)]\, \phi(y)\, dy \\
&\quad - \sigma_M \sigma_Y \delta_{MY} \int_{-\infty}^{q_Y} y^2 \cdot \Phi(y_Z)\, \phi(y)\, dy \\
&\quad + [\mu_Z - \mu_Y] \sigma_M \delta_{MY} \int_{-\infty}^{q_Y} y \cdot \Phi(y_Z)\, \phi(y)\, dy \\
&= \sigma_M \sigma_Z \delta_{MZ} \int_{-\infty}^{q_Y} \Phi(y_Z)\, \phi(y)\, dy \\
&\quad - \sigma_M \delta_{MZ} \int_{-\infty}^{q_Y} [\sigma_Y \cdot y + \mu_Y - \mu_Z] \phi(y_Z)\, \phi(y)\, dy \\
&\quad + (\sigma_Y \delta_{MZ}) \sigma_M \int_{-\infty}^{q_Y} y \phi(y_Z)\, \phi(y)\, dy \\
&\quad + (-\sigma_Z \delta_{MY}) \sigma_M \int_{-\infty}^{q_Y} y \phi(y_Z)\, \phi(y)\, dy \\
&\quad + [\mu_Z - \mu_Y] \sigma_M \delta_{MZ} \int_{-\infty}^{q_Y} [-\phi(y_Z)]\, \phi(y)\, dy \\
&\quad - \sigma_M \sigma_Y \delta_{MY} \int_{-\infty}^{q_Y} y^2 \cdot \Phi(y_Z)\, \phi(y)\, dy \\
&\quad + [\mu_Z - \mu_Y] \sigma_M \delta_{MY} \int_{-\infty}^{q_Y} y \cdot \Phi(y_Z)\, \phi(y)\, dy \\
&= \sigma_M \sigma_Z \delta_{MZ} \int_{-\infty}^{q_Y} \Phi(y_Z)\, \phi(y)\, dy - \sigma_Z \sigma_M \delta_{MY} \int_{-\infty}^{q_Y} y \phi(y_Z)\, \phi(y)\, dy \\
&\quad - \sigma_M \sigma_Y \delta_{MY} \int_{-\infty}^{q_Y} y^2 \cdot \Phi(y_Z)\, \phi(y)\, dy \\
&\quad + [\mu_Z - \mu_Y] \sigma_M \delta_{MY} \int_{-\infty}^{q_Y} y \cdot \Phi(y_Z)\, \phi(y)\, dy
\end{aligned}$$

Now we derive $\frac{\partial}{\partial Q}\text{Cov}(W(Q), M)$ and $\frac{\partial}{\partial Q}\text{Cov}(V(Q), M)$ for all three cases. Recalling that $\text{Cov}(W(Q), M) = \text{Cov}_1(W(Q), M) + \text{Cov}_2(W(Q), M) + \text{Cov}_3(W(Q), M) +$

$\mathrm{Cov}_4(W(Q), M)$, we have

$$\begin{aligned}
\mathrm{Cov}(W(Q), M) = \sigma_M \sigma_Z \delta_{MZ} \int_{-\infty}^{q_Y} \Phi(y_Z)\, \phi(y)\, dy - \sigma_Z \delta_{MY} \sigma_M \int_{-\infty}^{q_Y} y\phi(y_Z)\, \phi(y)\, dy \\
- \sigma_M \sigma_Y \delta_{MY} \int_{-\infty}^{q_Y} y^2 \cdot \Phi(y_Z)\, \phi(y)\, dy \\
+ [\mu_Z - \mu_Y] \sigma_M \delta_{MY} \int_{-\infty}^{q_Y} y \cdot \Phi(y_Z)\, \phi(y)\, dy \\
+ \sigma_M \sigma_Z \delta_{MZ} [1 - \Phi(q_Y)] \Phi(q_Z) + \sigma_M \sigma_Z \delta_{MY} \phi(q_Y)(-\phi(q_Z)) \\
+ \sigma_M \sigma_Z \delta_{MY} \phi(q_Y) \Phi(q_Z)(-q_Z)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial Q} \mathrm{Cov}(W(Q), M) = \mathrm{Cov}(Z, M) \left\{ \frac{\Phi(q_Z)\phi(q_Y)}{\sigma_Y} - \frac{\phi(q_Y)\Phi(q_Z)}{\sigma_Y} + \frac{[1 - \Phi(q_Y)]\phi(q_Z)}{\sigma_Z} \right\} \\
- \sigma_M \sigma_Z \delta_{MY} q_Y \phi(q_Z) \phi(q_Y)/\sigma_Y - \sigma_M \sigma_Y \delta_{MY} q_Y^2 \Phi(q_Z) \phi(q_Y)/\sigma_Y \\
+ \sigma_M \delta_{MY} [\mu_Z - \mu_Y] q_Y \Phi(q_Z) \phi(q_Y)/\sigma_Y \\
+ \sigma_M \sigma_Z \delta_{MY} [-q_Y \phi(q_Y)][-\phi(q_Z)]/\sigma_Y \\
+ \sigma_M \sigma_Z \delta_{MY} \phi(q_Y)[q_Z \phi(q_Z)]/\sigma_Z \\
+ \sigma_M \sigma_Z \delta_{MY} [-q_Y \phi(q_Y)] \Phi(q_Z)(-q_Z)/\sigma_Y \\
+ \sigma_M \sigma_Z \delta_{MY} \phi(q_Y) \phi(q_Z)(-q_Z)/\sigma_Z \\
+ \sigma_M \sigma_Z \delta_{MY} \phi(q_Y) \Phi(q_Z)(-1/\sigma_Z)
\end{aligned}$$

$$\frac{\partial}{\partial Q}\text{Cov}(W(Q), M) = \text{Cov}(Z, M)[1 - \Phi(q_Y)]\phi(q_Z)/\sigma_Z$$
$$+ \sigma_M \delta_{MY} \phi(q_Y)\Phi(q_Z) \cdot \frac{-[Q - \mu_Y]^2}{\sigma_Y^2}$$
$$+ \sigma_M \delta_{MY} \Phi(q_Z)\phi(q_Y) \cdot \frac{[\mu_Z - \mu_Y][Q - \mu_Y]}{\sigma_Y^2}$$
$$+ \sigma_M \sigma_Z \delta_{MY} \phi(q_Y)\Phi(q_Z) \cdot \frac{[Q - \mu_Z][Q - \mu_Y]}{\sigma_Y^2}$$
$$- \sigma_M \sigma_Z \delta_{MY} \phi(q_Y)\Phi(q_Z)/\sigma_Z$$
$$= \text{Cov}(Z, M)[1 - \Phi(q_Y)]\phi(q_Z)/\sigma_Z - \sigma_M \delta_{MY} \phi(q_Y)\Phi(q_Z)$$
$$= \text{Cov}(Z, M)[1 - \Phi(q_Y)]\phi(q_Z)/\sigma_Z - \text{Cov}(Y, M)\phi(q_Y)\Phi(q_Z)/\sigma_Y$$

In this case,

$$\frac{\partial}{\partial Q}\text{Cov}(V(Q), M) = r \cdot \frac{\partial}{\partial Q}\text{Cov}(U, M) + (r - s) \cdot \frac{\partial}{\partial Q}\text{Cov}(W, M)$$
$$= r\text{Cov}(Y, M)\phi(q_Y)/\sigma_Y + (r - s)\text{Cov}(Z, M)[1 - \Phi(q_Y)]\phi(q_Z)/\sigma_Z$$
$$- (r - s)\text{Cov}(Y, M)\phi(q_Y)\Phi(q_Z)/\sigma_Y$$
$$= \text{Cov}(Y, M)\phi(q_Y)/\sigma_Y \cdot [r - (r - s)\Phi(q_Z)]$$
$$+ (r - s)\text{Cov}(Z, M)[1 - \Phi(q_Y)]\phi(q_Z)/\sigma_Z$$

Knowing that $\frac{\partial}{\partial Q}\text{Cov}(D_1(Q), M) = -a(1+r_f)\text{Cov}(Y, M)\phi(q_Y)/\sigma_Y$, we have obtained all the results needed to prove Equation A.4. We have the first order derivative:

$$
\begin{aligned}
(1+r_f)\frac{\partial}{\partial Q}S(Q) &= \frac{\partial}{\partial Q}\mathbb{E}(D(Q)) - \Omega\frac{\partial}{\partial Q}\mathrm{Cov}(D_1(Q),M) - \Omega\frac{\partial}{\partial Q}\mathrm{Cov}(V(Q),M) \\
&= [1-\Phi(q_Y)][r-a(1+r_f)-(r-s)\Phi(q_Z)] \\
&\quad + \Omega a(1+r_f)\mathrm{Cov}(Y,M)\phi(q_Y)/\sigma_Y \\
&\quad - \Omega\mathrm{Cov}(Y,M)\phi(q_Y)/\sigma_Y\cdot[r-(r-s)\Phi(q_Z)] \\
&\quad - \Omega(r-s)\mathrm{Cov}(Z,M)[1-\Phi(q_Y)]\phi(q_Z)/\sigma_Z \qquad \text{(A.3)} \\
&= [1-\Phi(q_Y)]\{r-a(1+r_f)-(r-s)\Phi(q_Z) \\
&\quad - (r-s)\Omega\mathrm{Cov}(Z,M)\phi(q_Z)/\sigma_Z\} \\
&\quad - \Omega\mathrm{Cov}(Y,M)\phi(q_Y)/\sigma_Y\cdot[r-a(1+r_f)-(r-s)\Phi(q_Z)] \qquad \text{(A.4)}
\end{aligned}
$$

It follows from Equation A.4 and the FOC $\frac{\partial}{\partial Q}S(Q^*)=0$ that

$$
[1-\Phi(q_Y^*)]\{c_F-\Phi(q_Z^*)-s_R\delta_{MZ}\phi(q_Z^*)\}-s_R\delta_{MY}\phi(q_Y^*)\cdot[c_F-\Phi(q_Z^*)]=0.
$$

□

Proof of Corollary 2.3.2. (a) Without loss of generality, assume $\delta_{MZ}>0$ and thus $Q_a<Q_b$. Note that $(r-s)^{-1}(1+r_f)\frac{\partial}{\partial Q}S(Q)=[1-\Phi(q_Y)]\cdot H_a(Q)-s_R\delta_{MZ}\phi(q_Y)\cdot H_b(Q)$. It follows that $\frac{\partial}{\partial Q}S(Q)>0$ when $Q\le Q_a$ and $\frac{\partial}{\partial Q}S(Q)<0$ when $Q\ge Q_b$. Therefore, $Q^*\in(Q_a,Q_b)$. The case with $\delta_{MZ}<0$ can be proved in a similar manner and is therefore omitted.

(b) Without loss of generality, assume $\delta_{MZ}>0$ and thus $Q_a<Q_b$. Note that $(r-s)^{-1}(1+r_f)\frac{\partial}{\partial Q}S(Q)=[1-\Phi(q_Y)]\cdot H_a(Q)-s_R\delta_{MZ}\phi(q_Y)\cdot H_b(Q)$. It follows that $\frac{\partial}{\partial Q}S(Q)=[1-\Phi(q_Y)]H_a(Q)-\Omega\mathrm{Cov}(Y,M)\phi(q_Y)/\sigma_Y\cdot H_b(Q)<0$ if $Q_a\le Q\le Q_b$. Therefore, $Q^*\notin(Q_a,Q_b)$. The case with $\delta_{MZ}<0$ can be proved in a similar manner.

It follows from Proposition 2.3.5 and Corollary 2.3.2 (a) that if $\frac{\partial^2}{\partial Q^2}S(Q^*) < 0$, $Q^* < Q_a < Q_b$ when $\delta_{MZ} > 0$ and $\delta_{MY} > 0$, and $Q_b < Q_a < Q^*$ when $\delta_{MZ} < 0$ and $\delta_{MY} > 0$. □

Proof of Proposition 2.3.3. This proof is based on the implicit function theorem (Krantz and Parks, 2002). Let $\Pi = (r-s)^{-1}(1+r_f)\frac{\partial}{\partial Q}S(Q^*) = [1-\Phi(q_Y^*)][c_F - \Phi(q_Z^*) - s_R\delta_{MZ}\phi(q_Z^*)] - s_R\delta_{MY}\phi(q_Y^*)[c_F - \Phi(q_Z^*)] = 0$. Note that by assumption $\partial\Pi/\partial Q^* = (r-s)^{-1}(1+r_f)\frac{\partial^2}{\partial Q^2}S(Q^*) < 0$.

Since $\partial\Pi/\partial s_R = -[1-\Phi(q_Y^*)]\cdot\delta_{MZ}\phi(q_Z^*) - \delta_{MY}\phi(q_Y^*)[c_F - \Phi(q_Z^*)]$ , we have that $\frac{dQ^*}{ds_R} = -\frac{\partial\Pi/\partial s_R}{\partial\Pi/\partial Q^*}$. Note that $\Pi = [1-\Phi(q_Y^*)][c_F - \Phi(q_Z^*)] + s_R\cdot\partial\Pi/\partial s_R = 0$. We have that $\frac{dQ^*}{ds_R} < 0 \Leftrightarrow \partial\Pi/\partial s_R < 0 \Leftrightarrow c_F - \Phi(q_Z^*) > 0 \Leftrightarrow Q^* < Q_b$ and that $\frac{dQ^*}{ds_R} > 0 \Leftrightarrow \partial\Pi/\partial s_R > 0 \Leftrightarrow c_F - \Phi(q_Z^*) < 0 \Leftrightarrow Q^* > Q_b$. □

Proof of Proposition 2.3.4. This proof is based on the implicit function theorem (Krantz and Parks, 2002). Let $\Pi = (r-s)^{-1}(1+r_f)\frac{\partial}{\partial Q}S(Q^*) = [1-\Phi(q_Y^*)][c_F - \Phi(q_Z^*) - s_R\delta_{MZ}\phi(q_Z^*)] - s_R\delta_{MY}\phi(q_Y^*)[c_F - \Phi(q_Z^*)] = 0$. Since $\partial\Pi/\partial\delta_{MZ} = [1-\Phi(q_Y^*)]\cdot s_R\phi(q_Z^*) > 0$, we have that $\frac{dQ^*}{d\delta_{MZ}} = -\frac{\partial\Pi/\partial\delta_{MZ}}{\partial\Pi/\partial Q^*} < 0$. □

Proof of Proposition 2.3.5. This proof is based on the implicit function theorem (Krantz and Parks, 2002). Let $\Pi = (r-s)^{-1}(1+r_f)\frac{\partial}{\partial Q}S(Q^*) = [1-\Phi(q_Y^*)][c_F - \Phi(q_Z^*) - s_R\delta_{MZ}\phi(q_Z^*)] - s_R\delta_{MY}\phi(q_Y^*)[c_F - \Phi(q_Z^*)] = 0$. Since $\partial\Pi/\partial\delta_{MY} = -s_R\phi(q_Y^*)[c_F - \Phi(q_Z^*)]$, we have that when $Q^* < Q_b$, $[c_F - \Phi(q_Z^*)] > 0$ and $\frac{dQ^*}{d\delta_{MY}} = -\frac{\partial\Pi/\partial\delta_{MY}}{\partial\Pi/\partial Q^*} < 0$; when $Q^* > Q_b$, $[c_F - \Phi(q_Z^*)] < 0$ and $\frac{dQ^*}{d\delta_{MY}} = -\frac{\partial\Pi/\partial\delta_{MY}}{\partial\Pi/\partial Q^*} > 0$. Clearly, smaller size of $\delta_{MY}$ pulls $Q^*$ closer to $Q_b$. □

Proof of Proposition 2.3.6. This proof is based on the implicit function theorem (Krantz and Parks, 2002). Let $\Pi = (r-s)^{-1}(1+r_f)\frac{\partial}{\partial Q}S(Q^*) = [1-\Phi(q_Y^*)][c_F - \Phi(q_Z^*) - s_R\delta_{MZ}\phi(q_Z^*)] - s_R\delta_{MY}\phi(q_Y^*)[c_F - \Phi(q_Z^*)] = 0$. Note that by assumption

$\partial\Pi/\partial Q^* = (r-s)^{-1}(1+r_f)\frac{\partial^2}{\partial Q^2}S(Q^*) < 0$.

We have that $\partial\Pi/\partial\mu_Y = \phi(q_Y^*)/\sigma_Y[H_a(Q^*) - s_R\delta_{MY}q_Y^*H_b(Q^*)]$. Note that $\Pi(Q^*) = 0$, we have that

$$
\begin{aligned}
H_a(Q^*) - s_R\delta_{MY}q_Y^*H_b(Q^*) > 0 \Leftrightarrow & [1-\Phi(q_Y^*)]H_a(Q^*) \\
& - [1-\Phi(q_Y^*)]s_R\delta_{MY}q_Y^*H_b(Q^*) > 0 \\
\Leftrightarrow & \{[1-\Phi(q_Y^*)]q_Y^* - \phi(q_Y^*)\}s_R\delta_{MY}H_b(Q^*) < 0 \\
\Leftrightarrow & \{[1-\Phi(q_Y^*)]q_Y^* - \phi(q_Y^*)\}s_R\delta_{MY}(Q^* - Q_b) > 0 \\
\Leftrightarrow & \delta_{MY}(Q^* - Q_b) < 0 \\
\Leftrightarrow & \delta_{MY}\delta_{MZ} > 0
\end{aligned}
$$

, based on the result that $[1-\Phi(q_Y^*)](Y-\mu_Y)/\sigma_Y^2 < \phi(q_Y^*)/\sigma_Y$ (Feller, 1967, p. 175) and Corollary 2.3.2 (b). □

Proof of Proposition 2.3.7. This proof is based on the implicit function theorem (Krantz and Parks, 2002). Let $\Pi = (r-s)^{-1}(1+r_f)\frac{\partial}{\partial Q}S(Q^*) = [1-\Phi(q_Y^*)][c_F - \Phi(q_Z^*) - s_R\delta_{MZ}\phi(q_Z^*)] - s_R\delta_{MY}\phi(q_Y^*)[c_F - \Phi(q_Z^*)] = 0$. Note that by assumption $\partial\Pi/\partial Q^* = (r-s)^{-1}(1+r_f)\frac{\partial^2}{\partial Q^2}S(Q^*) < 0$.

Note that $\frac{dQ^*}{dc_F} = -\frac{\partial\Pi/\partial c_F}{\partial\Pi/\partial Q^*}$. Since $\partial\Pi/\partial c_F = 1 - \Phi(q_Y^*) - s_R\delta_{MY}\phi(q_Y^*)$ which equals zero when $Q^* = Q_{2b}$, we have that when $\delta_{MY} < 0$, $dQ^*/dc_F > 0$ for all values of $Q^*$; when $\delta_{MY} > 0$, $dQ^*/dc_F > 0$ when $Q^* < Q_{2b}$ and $dQ^*/dc_F < 0$ when $Q^* > Q_{2b}$. □

Proof of Lemma 2.5.1. We have the first-order derivative based on Equation A.1:

$$
\begin{aligned}
(1+r_f)\frac{\partial}{\partial Q}S(Q) = & [1-F(Q)]\{r - a(1+r_f) + d - (r-s+d)\Phi(q_Z) \\
& - (r-s+d)\Omega\text{Cov}(Z,M)\phi(q_Z)/\sigma_Z\}
\end{aligned}
\quad \text{(A.5)}
$$

We first present proof of Equation A.5.

Recall that $Y \sim F(Y)$, a generic distribution with $\text{Cov}(Y, M) = 0$. Also recall that $Z \sim G(Z) = \Phi\left(\frac{Z-\mu_Z}{\sigma_Z}\right)$, where $Z$ is normally distributed and correlated with the market return. Let $R(Q) = U(Q) - Q = \min\{Y - Q, 0\}$. We know from Anvari (1987) that $\text{Cov}(U(Q), M) = \text{Cov}(R(Q)+Q, M) = \text{Cov}(R(Q), M) = F(Q) \cdot \text{Cov}(Y, M) = 0$. It follows that $\text{Cov}(D_1(Q), M) = -a(1 + r_f)\text{Cov}(U(Q), M) = 0$.

To maximize $S(Q)$, we need to find the expression $\partial\text{Cov}(S(Q), M)/\partial Q$. We begin by calculating $\text{Cov}(V(Q), M)$. Define auxiliary function $W(Q)$ where

$$W(Q) = \begin{cases} -d(Z - Q) & \text{if } Y \geq Q \text{ and } Z \geq Q \\ -d(Z - Y) & \text{if } Y < Q \text{ and } Z \geq Y \\ (r - s)(Z - Q) & \text{if } Y \geq Q \text{ and } Z < Q \\ (r - s)(Z - Y) & \text{if } Y < Q \text{ and } Z < Y \end{cases}$$

Since $V(Q) = rU(Q) + W(Q)$, we have that $\text{Cov}(V(Q), M) = \text{Cov}(D_1(Q), M) + \text{Cov}(W(Q), M) = \text{Cov}(W(Q), M)$. We focus on finding the expression of $\text{Cov}(W(Q), M)$ where

$$\begin{aligned} \text{Cov}(W(Q), M) &= \mathbb{E}\{[W(Q) - \mathbb{E}(W(Q))][M - \mathbb{E}(M)]\} \\ &= \mathbb{E}\{W(Q)[M - \mathbb{E}(M)]\} - \mathbb{E}(W(Q))\mathbb{E}[M - \mathbb{E}(M)]\} \\ &= \mathbb{E}\{W(Q)[M - \mathbb{E}(M)]\}. \end{aligned}$$

Let

$$\begin{aligned}\mathbb{E}\{W(Q)[M-\mathbb{E}(M)]\} = \mathbb{E}\{[W(Q)][M-\mathbb{E}(M)]\}|_{Y\geq Q, Z\geq Q} \\ + \mathbb{E}\{[W(Q)][M-\mathbb{E}(M)]\}|_{Y<Q, Z\geq Y} \\ + \mathbb{E}\{W(Q)[M-\mathbb{E}(M)]\}|_{Y\geq Q, Z<Q} \\ + \mathbb{E}\{[W(Q)][M-\mathbb{E}(M)]\}|_{Y<Q, Z<Y},\end{aligned} \tag{A.6}$$

where each part corresponds to one scenario of $W(Q)$. Denote $y_Z = [Y - \mu_Z]/\sigma_Z$ and $q_Z = [Q - \mu_Z]/\sigma_Z$. Let $m$, $z$, and $y$ be the normalized standard normal variables of $M$, $Z$, and $Y$, respectively.

1) For the case of $Y \geq Q$ and $Z \geq Q$, we have

$$\begin{aligned}&\frac{\mathbb{E}\{[W(Q)][M-\mathbb{E}(M)]\}|_{Y\geq Q, Z\geq Q}}{-d} \\ &= \int_Q^{+\infty}\int_Q^{+\infty}\int_{-\infty}^{+\infty}[(Z-Q)]\sigma_M m \cdot f(m,Y,Z)\,dm\,dY\,dZ \\ &= \int_Q^{+\infty}\int_Q^{+\infty}(Z-Q)\left[\int_{-\infty}^{+\infty}\sigma_M m f(m|Y,Z)\,dm\right] f(Y,Z)\,dY\,dZ \\ &= \sigma_M\int_{q_Z}^{+\infty}\int_Q^{+\infty}(Z-Q)z\delta_{MZ}\, f(Y)\,dY\,\phi(z)\,dz \\ &= \delta_{MZ}\sigma_M\sigma_Z\int_{q_Z}^{+\infty}(z-q_Z)z[1-F(Q)]\,\phi(z)\,dz \\ &= \delta_{MZ}\sigma_M\sigma_Z[1-F(Q)]\int_{q_Z}^{+\infty}(z^2-q_Z\cdot z)\,\phi(z)\,dz \\ &= \delta_{MZ}\sigma_M\sigma_Z[1-F(Q)]\left\{\int_{q_Z}^{+\infty}z^2\,\phi(z)\,dz - q_Z\int_{q_Z}^{+\infty}z\,\phi(z)\,dz\right\} \\ &= \delta_{MZ}\sigma_M\sigma_Z[1-F(Q)]\Big\{[1-\Phi(q_Z)+q_Z\phi(q_Z)] - q_Z\phi(q_Z)\Big\} \\ &= \delta_{MZ}\sigma_M\sigma_Z[1-F(Q)][1-\Phi(q_Z)]\end{aligned}$$

2) For the case of $Y < Q$ and $Z \geq Y$, we have

$$
\begin{aligned}
&\frac{\mathbb{E}\{[W(Q)][M - \mathbb{E}(M)]\}|_{Y<Q,Z\geq Y}}{-d} \\
&= \int_0^Q \int_Y^{+\infty} \int_{-\infty}^{+\infty} [(Z - Y)]\sigma_M m \cdot f(m, Y, Z)\, dm\, dZ\, dY \\
&= \sigma_M \int_0^Q \int_Y^{+\infty} [Z - Y] \left[ \int_{-\infty}^{+\infty} m \cdot f(m|Y, Z)\, dm \right] f(Y, Z)\, dZ\, dY \\
&= \sigma_M \int_0^Q \int_{y_Z}^{+\infty} \sigma_Z [z - y_Z] z \delta_{MZ} \phi(z)\, dz\, f(Y)\, dY \\
&= \delta_{MZ}\sigma_M\sigma_Z \int_0^Q \int_{y_Z}^{+\infty} [z^2 - y_Z \cdot z]\phi(z)\, dz\, f(Y)\, dY \\
&= \delta_{MZ}\sigma_M\sigma_Z \int_0^Q \left\{ \int_{y_Z}^{+\infty} z^2\phi(z)\, dz - y_Z \int_{y_Z}^{+\infty} z\phi(z)\, dz \right\} f(Y)\, dY \\
&= \delta_{MZ}\sigma_M\sigma_Z \int_0^Q \{[1 - \Phi(Y_Z) + y_Z\phi(Y_Z)] - y_Z\phi(Y_Z)\}\ f(Y)\, dY \\
&= \delta_{MZ}\sigma_M\sigma_Z \int_0^Q [1 - \Phi(Y_Z)]\, f(Y)\, dY
\end{aligned}
$$

3) For the case of $Y \geq Q$ and $Z < Q$, we have

$$\begin{aligned}
&\frac{\mathbb{E}\{W(Q)[M - \mathbb{E}(M)]\}|_{Y \geq Q, Z < Q}}{r - s} \\
&= \int_0^Q \int_Q^{+\infty} \int_{-\infty}^{+\infty} [(Z - Q)]\sigma_M m \cdot f(m, Y, Z)\, dm\, dY\, dZ \\
&= \int_0^Q \int_Q^{+\infty} (Z - Q) \left[ \int_{-\infty}^{+\infty} \sigma_M m \cdot f(m|Y, Z)\, dm \right] f(Y, Z)\, dY\, dZ \\
&= \sigma_M \int_{-\infty}^{q_Z} \int_Q^{+\infty} [\sigma_Z(z - q_Z)] z \delta_{MZ}\, f(Y)\, dY\, \phi(z)\, dz \\
&= \delta_{MZ}\sigma_M\sigma_Z \int_{-\infty}^{q_Z} [(z - q_Z)] z [1 - F(Q)]\, \phi(z)\, dz \\
&= \delta_{MZ}\sigma_M\sigma_Z [1 - F(Q)] \left\{ \int_{-\infty}^{q_Z} z^2\, \phi(z)\, dz + \int_{-\infty}^{q_Z} (-q_Z) z\, \phi(z)\, dz \right\} \\
&= \delta_{MZ}\sigma_M\sigma_Z [1 - F(Q)] \Big\{ [\Phi(q_Z) - q_Z\phi(q_Z)] + (-q_Z)(-\phi(q_Z)) \Big\} \\
&= \sigma_M\sigma_Z\delta_{MZ} [1 - F(Q)] \Phi(q_Z)
\end{aligned}$$

4) For the case of $Y < Q$ and $Z < Y$, we have

$$\begin{aligned}
&\frac{\mathbb{E}\{[W(Q)][M - \mathbb{E}(M)]\}|_{Y < Q, Z < Y}}{r - s} \\
&= \int_0^Q \int_0^Y \int_{-\infty}^{+\infty} [(Z - Y)]\sigma_M m \cdot f(m, Y, Z)\, dm\, dZ\, dY \\
&= \sigma_M \int_0^Q \int_0^Y [Z - Y] \left[ \int_{-\infty}^{+\infty} m \cdot f(m|Y, Z)\, dm \right] f(Y, Z)\, dZ\, dY \\
&= \sigma_M \int_0^Q \int_{-\infty}^{y_Z} \sigma_Z(z - y_Z) \cdot z\delta_{MZ}\, \phi(z)\, dz\, f(Y)\, dY \\
&= \delta_{MZ}\sigma_M\sigma_Z \int_0^Q \left\{ \int_{-\infty}^{y_Z} z^2\, \phi(z)\, dz\ - \int_{-\infty}^{y_Z} y_Z z\, \phi(z)\, dz \right\} f(Y)\, dY \\
&= \sigma_M\sigma_Z\delta_{MZ} \int_0^Q \{[\Phi(Y_Z) - y_Z\phi(Y_Z)] - y_Z[-\phi(Y_Z)]\}\, f(Y)\, dY \\
&= \sigma_M\sigma_Z\delta_{MZ} \int_0^Q \Phi(Y_Z)\, f(Y)\, dY
\end{aligned}$$

Now we derive $\frac{\partial}{\partial Q}\text{Cov}(W(Q), M)$ and $\frac{\partial}{\partial Q}\text{Cov}(V(Q), M)$. Recalling Equation A.6, we have

$$\begin{aligned}\text{Cov}(W(Q), M) = \text{Cov}(Z, M)\Big\{ &-d[1-F(Q)][1-\Phi(q_Z)] - d\int_0^Q [1-\Phi(Y_Z)]\, f(Y)\, dY \\ &+ (r-s)\int_0^Q \Phi(Y_Z)\, f(Y)\, dY + (r-s)[1-F(Q)]\Phi(q_Z)\Big\}\end{aligned}$$

and the first-order derivative is

$$\begin{aligned}\frac{\partial}{\partial Q}\text{Cov}(W(Q), M) &= \text{Cov}(Z, M)\Big\{d\{f(Q)[1-\Phi(q_Z)] + [1-F(Q)][\phi(q_Z)/\sigma_Z] \\ &\quad - [1-\Phi(q_Z)]f(Q)\} + (r-s)\{\Phi(q_Z)f(Q) - f(Q)\Phi(q_Z) \\ &\quad + [1-F(Q)]\phi(q_Z)/\sigma_Z\Big\} \\ &= \text{Cov}(Z, M)(r-s+d)[1-F(Q)]\phi(q_Z)/\sigma_Z\end{aligned}$$

We have that $\mathbb{E}(U(Q)) = [1-F(Q)]\cdot Q + \int_0^Q Y\, dF(Y)$, $\frac{\partial}{\partial Q}\mathbb{E}(U(Q)) = 1 - F(Q) -$

$f(Q)Q + Qf(Q) = 1 - F(Q)$, and $\mathbb{E}(D_1(Q)) = (1 + r_f)[-a \cdot \mathbb{E}(U(Q))]$.

$$\begin{aligned}
\mathbb{E}(W(Q)) = -d\Bigg\{ & \int_Q^{+\infty} \int_Q^{+\infty} (Z-Q)g(Z)dZf(Y)dY \\
& + \int_0^Q \int_Y^{+\infty} (Z-Y)g(Z)dZf(Y)dY \Bigg\} \\
& + (r-s)\Bigg\{ \int_Q^{+\infty} \int_0^Q (Z-Q)g(Z)dZf(Y)dY \\
& + \int_0^Q \int_0^Y (Z-Y)g(Z)dZf(Y)dY \Bigg\} \\
= -d\Bigg\{ & [1-F(Q)]\Bigg[ \int_Q^{+\infty} Zg(Z)dZ - Q(1-G(Q)) \Bigg] \\
& + \int_0^Q \left[ \int_Y^{+\infty} Zg(Z)dZ - Y(1-G(Y)) \right] f(Y)dY \Bigg\} \\
& + (r-s)\left\{ [1-F(Q)]\Bigg[ \int_0^Q Zg(Z)dZ - QG(Q) \Bigg] \right. \\
& \left. + \int_0^Q \left[ \int_0^Y Zg(Z)dZ - YG(Y) \right] f(Y)dY \right\}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial Q}\mathbb{E}(W(Q)) = -d \cdot \Bigg\{ & -f(Q)\Bigg[ \int_Q^{+\infty} Zg(Z)dZ - Q(1-G(Q)) \Bigg] \\
& + [1-F(Q)][-Qg(Q) - 1 + G(Q) + Qg(Q)] \Bigg\} \\
& - d\left[ \int_Q^{+\infty} Zg(Z)dZ - Q(1-G(Q)) \right] f(Q) \\
& + (r-s)\Bigg\{ -f(Q)\left[ \int_0^Q Zg(Z)dZ - \int_0^Q Qg(Z)dZ \right] \\
& + [1-F(Q)][Qg(Q) - Qg(Q) - G(Q)] \Bigg\} \\
& + (r-s)f(Q)\left[ \int_0^Q Zg(Z)dZ - \int_0^Q Qg(Z)dZ \right] \\
= [1-F(Q)]&[d - (r-s+d)G(Q)].
\end{aligned}$$

Thus, we have that $\frac{\partial}{\partial Q}\mathbb{E}(D(Q)) = \frac{\partial}{\partial Q}\mathbb{E}(D_1(Q))+\frac{\partial}{\partial Q}\mathbb{E}(V(Q)) = -a(1+r_f)\frac{\partial}{\partial Q}\mathbb{E}(U(Q))+ r \cdot \frac{\partial}{\partial Q}\mathbb{E}(U(Q)) + \frac{\partial}{\partial Q}\mathbb{E}(W(Q)) = [1-F(Q)][r-a(1+r_f)+d-(r-s+d)G(Q)]$. It follows that

$$
\begin{aligned}
(1+r_f)\frac{\partial S(Q)}{\partial Q} &= \left[\frac{\partial}{\partial Q}\mathbb{E}(D_1(Q)) - \Omega\frac{\partial}{\partial Q}\mathrm{Cov}(D_1(Q), M)\right] \\
&\quad + \left[\frac{\partial}{\partial Q}\mathbb{E}(V(Q)) - \Omega\frac{\partial}{\partial Q}\mathrm{Cov}(V(Q), M)\right] \\
&= \frac{\partial}{\partial Q}\mathbb{E}(D(Q)) - \Omega\frac{\partial}{\partial Q}\mathrm{Cov}(D_1(Q), M) - \Omega\frac{\partial}{\partial Q}\mathrm{Cov}(W(Q), M) \\
&= [1-F(Q)][r-a(1+r_f)+d-(r-s+d)\Phi(q_Z)] \\
&\quad - \Omega(r-s+d)[1-F(Q)]\mathrm{Cov}(Z, M)\phi(q_Z)/\sigma_Z \\
&= [1-F(Q)]\{r-a(1+r_f)+d-(r-s+d) \\
&\quad \cdot [\Phi(q_Z)+\Omega\mathrm{Cov}(Z, M)\phi(q_Z)/\sigma_Z]\}
\end{aligned}
$$

After finishing the proof of Equation A.5, we can easily arrive at Lemma 2.5.1. Note that $q_Z^* = [Q^* - \mu_Z]/\sigma_Z$. It follows from the FOC $\frac{\partial}{\partial Q}S(Q^*) = 0$ and the fact that $1 - F(Q^*) > 0$. Note that $\frac{\partial}{\partial Q}S(Q^*) = 0$ when $Q^* \geq Y_{max}$. □

PROOF OF LEMMA 2.5.2. Noting that $Q^*$ satisfies Equation 2.4, the second-order derivative at $Q = Q^*$ becomes:

$$(1+r_f)\frac{\partial^2}{\partial Q^2}S(Q^*) = -f(Q^*)\Bigg\{r - a(1+r_f) + d - (r-s+d)\Phi\left(\frac{Q^*-\mu_Z}{\sigma_Z}\right)$$
$$- (r-s+d)\Omega\text{Cov}(Z,M)\phi\left(\frac{Q^*-\mu_Z}{\sigma_Z}\right)/\sigma_Z\Bigg\}$$
$$+ [1-F(Q^*)](r-s+d)\phi(q_Z^*)/\sigma_Z \tag{A.7}$$
$$\cdot\left[-1+\Omega\text{Cov}(Z,M)\frac{Q^*-\mu_Z}{\sigma_Z^2}\right]$$
$$= [1-F(Q^*)](r-s+d)\phi(q_Z^*)/\sigma_Z\cdot\left[-1+\Omega\text{Cov}(Z,M)\frac{Q^*-\mu_Z}{\sigma_Z^2}\right] \tag{A.8}$$

We continue to examine the second-order condition at $Q = Q^*$. The following analysis is similar to that of Chung (1990), but the random capacity introduces additional complexities. From Equation 2.4, we have that

$$(r-s+d)\Omega\text{Cov}(Z,M) = [r-(1-r_f)a+d-(r-s+d)G(Q^*)]/g(Q^*) \tag{A.9}$$

Substituting Equation A.9 into the second order condition, we obtain

$$(1+r_f)\frac{\partial^2}{\partial Q^2}S(Q^*) = -(r-s+d)[1-F(Q^*)]\left\{g(Q^*)-[Q^*-\mu_Z][c_F-G(Q^*)]/\sigma_Z^2\right\} \tag{A.10}$$

Note that $c_F < 1$.

Case 1: $\text{Cov}(Z,M) > 0$. When $Q^* \leq \mu_Z$, a sufficient condition for $\frac{\partial^2}{\partial Q^2}S(Q^*) < 0$ is $c_F > G(Q^*)$, and from Equation 2.4 we have $c_F > G(Q^*)$ since $\text{Cov}(Z,M) > 0$ and hence $\frac{\partial^2}{\partial Q^2}S(Q^*) < 0$. If $Q^* > \mu_Z$, we utilize the following property of the normal distribution (Feller, 1967, p. 175): $g(x) > [x-\mu_Z][1-G(x)]/\sigma_Z^2$. Since $c_F < 1$, we have that $g(Q^*) > [Q^*-\mu_Z][c_F-G(Q^*)]$ and hence $\frac{\partial^2}{\partial Q^2}S(Q^*) < 0$. Thus for the case $\text{Cov}(Z, M) > 0$, we always have $\frac{\partial^2}{\partial Q^2} S(Q^*) < 0$.

Case 2: $\text{Cov}(Z, M) < 0$. Let $Q_d = \mu_Z + \sigma_Z^2/[\Omega\text{Cov}(Z, M)]$. When $Q^* > \mu_Z$, then from Equation A.10 we know $\frac{\partial^2}{\partial Q^2} S(Q^*) < 0$ since we have $c_F < G(Q^*)$ from Equation 2.4. When $Q^* = \mu_Z$, it can be shown that $(1 + r_f)\frac{\partial^2}{\partial Q^2} S(Q^*) = -(r - s + d)[1 - F(Q^*)]g(Q^*) < 0$. When $Q^* < \mu_Z$, then it must be that $Q_d < Q^* \leq \mu_Z$. To have $\frac{\partial^2}{\partial Q^2} S(Q^*) < 0$, it must be that $Q^* > Q_d$, which can be shown below. From Equation A.10, we have $g(Q^*) - [Q^* - \mu_Z][c_F - G(Q^*)]/\sigma_Z^2 > 0 \Leftrightarrow \sigma_Z^2 g(Q^*) > [Q^* - \mu_Z][c_F - G(Q^*)] \Leftrightarrow G(Q^*) + \sigma_Z^2/[Q^* - \mu_Z] \cdot g(Q^*) < c_F = \frac{r - a(1+r_f)+d}{r-s+d}$.

Comparing to Equation 2.4, we have that $G(Q^*) + \sigma_Z^2/[Q^* - \mu_Z] \cdot g(Q^*) < c_F \Leftrightarrow \Omega\text{Cov}(Z, M) > \sigma_Z^2/[Q^* - \mu_Z] \Leftrightarrow Q^* > \sigma_Z^2/[\Omega\text{Cov}(Z, M)] + \mu_Z = Q_d$. If $Q^* < Q_d$, we have that $\frac{\partial^2}{\partial Q^2} S(Q^*) > 0$. Since $\frac{\partial}{\partial Q} S(Q)|_{Q=0} = [r - a(1 + r_f) + d]/(1 + r_f) > 0$, $\frac{\partial}{\partial Q} S(Q^*) = 0$, and $\frac{\partial^2}{\partial Q^2} S(Q^*) > 0$, by the extreme value theorem, there must exist $0 < Q^{**} < Q^*$ such that $\frac{\partial}{\partial Q} S(Q^{**}) = 0$ and $Q^{**}$ is a local interior maximizer of $S(Q)$ in interval $[0, Q^*]$; it follows from the property of a local interior maxima of a differentiable function that $\frac{\partial^2}{\partial Q^2} S(Q^{**}) \leq 0$, which conflicts with $\frac{\partial}{\partial Q} S(Q^{**}) = 0 \Rightarrow \frac{\partial^2}{\partial Q^2} S(Q^{**}) > 0$ since $Q^{**} < Q_d$. It follows that $Q^* > Q_d$, and we have that $\frac{\partial^2}{\partial Q^2} S(Q) < 0$ and the second-order sufficient condition is satisfied.

Case 3: $\text{Cov}(Z, M) = 0$. In this case, $\frac{\partial^2}{\partial Q^2} S(Q^*) = (1+r_f)^{-1}\sigma_Z^{-1}[1-F(Q^*)](r-s+d)\phi(q_Z) < 0$ for any given $Q^*$. Thus, $\frac{\partial^2}{\partial Q^2} S(Q^*) < 0$ in all the three cases, regardless of the sign of $\text{Cov}(Z, M)$. □

Proof of Theorem 2.5.3. Based on Lemma 2.5.1 and Lemma 2.5.2, suppose there exist two stationary points satisfying both FOC and SOC denoted as $Q_a$ and $Q_b$, and no other stationary point satisfying both FOC and SOC exists between $Q_a$ and $Q_b$. Without loss of generality, let $Q_a < Q_b$, from FOC and SOC we have that there exists $\epsilon_a, \epsilon_b > 0$ such that $\frac{\partial}{\partial Q} S(Q_a + \epsilon_a) < 0$ and $\frac{\partial}{\partial Q} S(Q_b - \epsilon_b) > 0$, where $\epsilon_a, \epsilon_b$ are very small. It follows from the continuity of $\frac{\partial}{\partial Q}S(Q)$ that there exists $Q_a + \epsilon < Q_c < Q_b - \epsilon$ such that $\frac{\partial}{\partial Q}S(Q_c) = 0$, where $Q_c$ satisfies FOC and should satisfy SOC based on Lemma 2.5.2. However, $Q_c$ satisfying SOC conflicts with the assumption that no other stationary point satisfying both FOC and SOC exists between $Q_a$ and $Q_b$. Therefore, $Q^*$ is single-valued. Note that $\frac{\partial}{\partial Q}S(Q)|_{Q=0} > 0$ and $\lim_{Q\to+\infty} \frac{\partial}{\partial Q}S(Q) < 0$. As $S(Q)$ is a single-variable function, satisfying both the first-order condition and the second-order condition ensures that $Q^*$ is the optimal order quantity and $S(Q^*)$ is the global maximum. $Q^*$ remains optimal when $Q^* \geq Y_{max}$. □

Proof of Corollary 2.5.4. This proof is based on the implicit function theorem (Chiang, 1984, p. 208). Denote $\Pi = \Phi(q_Z^*) + s_R\, \delta_{MZ}\, \phi(q_Z^*) - c_F = 0$. We have that $\partial\Pi/\partial Q^* = -\frac{1+r_f}{(r-s+d)[1-F(Q^*)]}\frac{\partial^2}{\partial Q^2}S(Q^*) > 0$ based on Equation A.8 and Lemma 2.5.2. Note that $\partial\Pi/\partial Q^* = \phi(q_Z^*)(1 - s_R\, \delta_{MZ}\, q_Z^*)/\sigma_Z$.

(a) Since $\partial\Pi/\partial s_R = \delta_{MZ}\, \phi(q_Z^*)$, we have that $\frac{dQ^*}{ds_R} = -\frac{\partial\Pi/\partial s_R}{\partial\Pi/\partial Q^*} = \frac{\delta_{MZ}\phi(q_Z^*)}{-\partial\Pi/\partial Q^*}$. We have that $Q^* < Q_C$ and $\frac{dQ^*}{ds_R} < 0$ when $\delta_{MZ} > 0$, and that $Q^* > Q_C$ and $\frac{dQ^*}{ds_R} > 0$ when $\delta_{MZ} < 0$, meaning that $Q^*$ moves further away from $Q_C$ as $s_R$ increases in either case.

(b) Since $\partial\Pi/\partial\delta_{MZ} = s_R\, \phi(q_Z^*)$, we have that $\frac{dQ^*}{d\delta_{MZ}} = -\frac{\partial\Pi/\partial\delta_{MZ}}{\partial\Pi/\partial Q^*} = \frac{s_R\, \phi(q_Z^*)}{-\partial\Pi/\partial Q^*} < 0$. It follows that when $\delta_{MZ} > 0$, $Q^* < Q_C$ and $\frac{dQ^*}{d\delta_{MZ}} < 0$, and that when $\delta_{MZ} < 0$, $Q^* > Q_C$ and $\frac{dQ^*}{d\delta_{MZ}} < 0$, meaning that $Q^*$ moves further away from $Q_C$ as $|\delta_{MZ}|$ increases in either case.

(c) Since $\partial\Pi/\partial\sigma_Z = \phi(q_Z^*)(-q_Z^*/\sigma_Z) - s_R\, \delta_{MZ}\, q_Z^*\phi(q_Z^*)(-q_Z^*/\sigma_Z)$, we have that $\frac{dQ^*}{d\sigma_Z} = -\frac{\partial\Pi/\partial\sigma_Z}{\partial\Pi/\partial Q^*} = q_Z^* = (Q^* - \mu_Z)/\sigma_Z$. Similarly, $\frac{dQ_C}{d\sigma_Z} = (Q_C - \mu_Z)/\sigma_Z$. It follows that when $Q^* > Q_d$, $\frac{d(Q^*-Q_C)}{d\sigma_Z} = \frac{dQ^*}{d\sigma_Z} - \frac{dQ_d}{d\sigma_Z} = (Q^* - Q_C)/\sigma_Z > 0$; when $Q^* < Q_C$,

$\frac{d(Q^*-Q_C)}{d\sigma_Z} = \frac{dQ^*}{d\sigma_Z} - \frac{dQ_C}{d\sigma_Z} = (Q^* - Q_C)/\sigma_Z < 0$; in both cases $|Q^* - Q_C|$ increases in $\sigma_Z$.

(d) Since $\partial\Pi/\partial\mu_Z = \phi(q_Z^*)\sigma_Z^{-1}(-1 + s_R\delta_{MZ}q_Z^*) = -\partial\Pi/\partial Q^*$, we have that $\frac{dQ^*}{d\mu_Z} = -\frac{\partial\Pi/\partial\mu_Z}{\partial\Pi/\partial Q^*} = 1 = \frac{dQ_C}{d\mu_Z}$, and thus both $Q^*$ and $Q_C$ increase with $|Q^* - Q_C|$ remaining the same.

(e) Since $\partial\Pi/\partial c_F = -1$, we have that $\frac{dQ^*}{dc_F} = -\frac{\partial\Pi/\partial c_F}{\partial\Pi/\partial Q^*} = \frac{1}{\partial\Pi/\partial Q^*} > 0$ and that $\frac{dQ_C}{dc_F} = \frac{\sigma_Z}{\phi(q_Z^C)} > 0$, where $q_Z^C = (Q_C - \mu_Z)/\sigma_Z$.

□

Proof of Proposition 2.5.5. 1. Preliminaries:

Based on Equation 2.5, we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}Y}P(Y) &= [r - a(1+r_f) + d] - (r - s + d)\int_0^Y g(Z)dZ \\
&\quad - (r - s + d)\delta_{MZ}s_R\phi\left(\frac{Y-\mu_Z}{\sigma_Z}\right)/\sigma_Z \\
&= [r - a(1+r_f) + d] - (r - s + d)\left[\Phi\left(\frac{Y-\mu_Z}{\sigma_Z}\right) + \delta_{MZ}s_R\phi\left(\frac{Y-\mu_Z}{\sigma_Z}\right)\right] \\
&= (r - s + d)\left[c_F - \Phi\left(\frac{Y-\mu_Z}{\sigma_Z}\right) - \delta_{MZ}s_R\phi\left(\frac{Y-\mu_Z}{\sigma_Z}\right)\right]
\end{aligned} \tag{A.11}$$

is negative when $Y \in [Y_{min}, Q^*]$ based on Lemmas 2.5.1 and 2.5.2, It follows that $P(Y)$ is an increasing function when $Y \in (Y_{min}, Q^*)$ until $\frac{d}{dY}P(Y) = 0$ at $Y = Q^*$, and hence

$$\overline{P}(Y) = \begin{cases} P(Y) & \text{if } Y < Q^* \\ P(Q^*) & \text{if } Y \geq Q^* \end{cases}$$

is an increasing function when $Y \in [Y_{min}, Q^*]$ and non-decreasing otherwise.

Note that $(1+r_f)S(Q^*) = P(Q^*)[1-F(Q^*)] + \int_0^{Q^*} P(Y)f(Y)dy$. We explore when $P(Y)$ is concave.

If $\delta_{MZ} = 0$, the firm aims to maximize its expected profit, and we have that $\frac{\mathrm{d}^2}{\mathrm{d}Y^2}P(Y) < 0$ when $Y \in \{Y_{min}, Q^*\}$, meaning that $P(Y)$ is concave.

If $\delta_{MZ} \neq 0$, we have that

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}Y^2}P(Y) &= -(r-s+d)\left[\phi\left(\frac{Y-\mu_Z}{\sigma_Z}\right)/\sigma_Z + \delta_{MZ}s_R\cdot\left(-\frac{Y-\mu_Z}{\sigma_Z^2}\right)\phi\left(\frac{Y-\mu_Z}{\sigma_Z}\right)\right]\\ &= -(r-s+d)\phi\left(\frac{Y-\mu_Z}{\sigma_Z}\right)/\sigma_Z\cdot\left[1-\delta_{MZ}s_R\cdot\left(\frac{Y-\mu_Z}{\sigma_Z}\right)\right].\end{aligned}$$

We cannot prove that $P(Y)$ is concave everywhere. However, note that Equation A.8 in the Proof of Lemma 2 shows that

$$(1+r_f)\frac{\partial^2}{\partial Q^2}S(Q^*) = -[1-F(Q^*)](r-s+d)\phi(q_Z)/\sigma_Z\cdot\left[1-\delta_{MZ}s_R\frac{Q^*-\mu_Z}{\sigma_Z}\right] < 0.$$

When $\delta_{MZ} > 0$, we have from $Y < Q^*$ that $1-\delta_{MZ}s_R\cdot\left(\frac{Y-\mu_Z}{\sigma_Z}\right) > 1-\delta_{MZ}s_R\cdot\left(\frac{Q^*-\mu_Z}{\sigma_Z}\right) > 0$ and hence $\frac{\mathrm{d}^2}{\mathrm{d}Y^2}P(Y) < 0$.

When $\delta_{MZ} < 0$, it follows that if $1-\delta_{MZ}s_R\cdot\left(\frac{Y-\mu_Z}{\sigma_Z}\right) > 0$ when $Y \in (Y_{min}, Q^*)$, it is guaranteed that $\frac{\mathrm{d}^2}{\mathrm{d}Y^2}P(Y) < 0$ when $Y \in (Y_{min}, Q^*)$. We arrive at a sufficient condition for $\frac{\mathrm{d}^2}{\mathrm{d}Y^2}P(Y) < 0$ when $\delta_{MZ} < 0$ and $Y \in \{Y_{min}, Q^*\}$:

$$Y_{min} > \mu_Z + \frac{\sigma_Z}{\delta_{MZ}s_R}.$$

which is Result (c).

We proceed to prove several properties:

2. Result (a)

Since $P_c(Y)$ is increasing when $Y \in (Y_{min}, Q^*)$, we can proceed to show one property of $S(Q^*)$.

We have that $\mathbb{E}(P)|_{\Delta Y=0} = \int_0^{+\infty} \overline{P}(Y)f(Y)dY$. Shifting the p.d.f. of the random capacity $f(Y)$ by $\Delta Y > 0$, we have

$$\mathbb{E}(P) = \int_0^{+\infty} \overline{P}(Y)f(Y-\Delta Y)dY = \int_0^{+\infty} \overline{P}(Y+\Delta Y)f(Y)dY.$$

It follows that

$$\frac{\mathrm{d}\mathbb{E}(P)}{\mathrm{d}\Delta Y} = \int_0^{+\infty} \frac{\partial}{\partial \Delta Y}\overline{P}(Y+\Delta Y)f(Y)dY = \int_0^{+\infty} \overline{P}'(Y+\Delta Y)f(Y)dY > 0.$$

In other words, shifting $f(Y)$ to the right improves the expected profit $\mathbb{E}(P)$. The proof of part (a) is based on the fact that $\overline{P}(Y)$ is a non-decreasing function and an increasing function at certain intervals.

When $P(Y)$ is concave, we proceed to show additional properties of $S(Q^*)$ in Proposition 2.5.5(b).

3. Result (b)(i)

It follows from Result (a) that

$$\frac{\mathrm{d}^2\mathbb{E}(P)}{\mathrm{d}(\Delta Y)^2} = \int_0^{+\infty} \overline{P}''(Y+\Delta Y)f(Y)dY.$$

Note that $\overline{P}''(Y+\Delta Y) = 0$ when $Y+\Delta Y \geq x^*$ and $\overline{P}''(Y+\Delta Y) < 0$ when $Y+\Delta Y < x^*$. It follows that

$$\frac{\mathrm{d}^2\mathbb{E}(P)}{\mathrm{d}(\Delta Y)^2} = \int_0^{x^*-\Delta Y} \overline{P}''(Y+\Delta Y)f(Y)dY < 0.$$

The proof of (b) is based on the fact that $\overline{P}(Y)$ is a *concave*, non-decreasing function and an increasing function at certain intervals. Also note that $\lim_{\Delta Y \to +\infty} \mathbb{E}(P) =$

$\int_0^{+\infty} \overline{P}(x^*) f(Y) dY = \overline{P}(x^*)$.

4. Result (b)(ii)

Recall that $\widetilde{\mathbb{E}}(P) = \int_0^{+\infty} \overline{P}(Y)\widetilde{f}(Y)dY = \int_0^{+\infty} \overline{P}(Y) \cdot bf\left[\mu_Y + b(Y-\mu_Y)\right]dY$. Let $u = \mu_Y + b(Y - \mu_Y)$ and hence $y = (u-\mu_Y)/b + \mu_Y$. Denote $L(u) = \overline{P}\left[(u-\mu_Y)/b + \mu_Y\right]$.

(i) Using the first mean value theorem for integration (Sahoo and Riedel, 1998, p.208), we have that

$$\widetilde{\mathbb{E}}(P) = \int_0^{+\infty} \overline{P}\left[(u-\mu_Y)/b + \mu_Y\right] f(u)du$$

It follows that

$$\begin{aligned}
\frac{\mathrm{d}\widetilde{\mathbb{E}}(P)}{\mathrm{d}b} &= \int_0^{+\infty} \frac{\partial}{\partial b}\overline{P}\left[(u-\mu_Y)/b + \mu_Y\right] f(u)du \\
&= \int_0^{+\infty} \overline{P}'\left[(u-\mu_Y)/b + \mu_Y\right] \cdot [-(u-\mu_Y)/b^2] f(u)du \\
&= \int_0^{\mu_Y} L'(u) \cdot [-(u-\mu_Y)/b^2] f(u)du + \int_{\mu_Y}^{+\infty} L'(u) \cdot [-(u-\mu_Y)/b^2] f(u)du \\
&= L'(u_1)\int_0^{\mu_Y} [-(u-\mu_Y)/b^2] f(u)du + L'(u_2)\int_{\mu_Y}^{+\infty} [-(u-\mu_Y)/b^2] f(u)du
\end{aligned}$$

where $u_1 \in (0, \mu_Y)$ and $u_2 \in (\mu_Y, +\infty)$. Since $\overline{P}(\cdot)$ is non-decreasing and concave, $L(\cdot)$ is also non-decreasing and concave. It follows that $L'(u_1) > L'(u_2) > 0$. Note that $\int_0^{\mu_Y} \cdot [-(u-\mu_Y)/b^2] f(u)du > 0$; also note that $\int_0^{\mu_Y} [-(u-\mu_Y)/b^2] f(u)du + \int_{\mu_Y}^{+\infty} [-(u-\mu_Y)/b^2] f(u)du = -b^{-2}\int_0^{+\infty} (u-\mu_Y) f(u)du = 0$ since $\mu_Y = E[f(\cdot)]$. We conclude that $\frac{\mathrm{d}\widetilde{\mathbb{E}}(P)}{\mathrm{d}b} > 0$.

5. Result (b)(iii)

(ii) Moreover, we have that

$$\begin{aligned}
\frac{\mathrm{d}^2\widetilde{\mathbb{E}}(P)}{\mathrm{d}b^2} &= \int_0^{+\infty} \frac{\partial^2}{\partial b^2}\overline{P}\left[(u-\mu_Y)/b+\mu_Y\right] f(u)du \\
&= \int_0^{+\infty} \frac{\partial}{\partial b}\left\{\overline{P}'\left[(u-\mu_Y)/b+\mu_Y\right]\cdot\left[-(u-\mu_Y)/b^2\right]\right\} f(u)du \\
&= \int_0^{+\infty} \left\{\overline{P}''\left[(u-\mu_Y)/b+\mu_Y\right]\cdot\frac{(u-\mu_Y)^2}{b^4}\right. \\
&\quad \left. +\,\overline{P}'\left[(u-\mu_Y)/b+\mu_Y\right]\cdot\frac{2(u-\mu_Y)}{b^3}\right\}\cdot f(u)du \\
&= \int_0^{+\infty} \left\{\overline{P}''\left[(u-\mu_Y)/b+\mu_Y\right]\cdot\frac{(u-\mu_Y)^2}{b^4}\right\} f(u)du - \frac{2}{a}\cdot\frac{\mathrm{d}\widetilde{\mathbb{E}}(P)}{\mathrm{d}b} \\
&< 0
\end{aligned}$$

Compared to the original distribution $f(Y)$, $\widetilde{f}(Y)$ is unchanged when $b=1$ and shrinks towards the mean when $b>1$. Denote $\sigma^2 = \int_0^\infty (Y-\mu_Y)^2 f(Y)dY$, we have that $\widetilde{\sigma}^2 = \int_0^\infty (Y-\mu_Y)^2 bf\left[\mu_Y + b(Y-\mu_Y)\right]dY = \int_0^\infty b^{-2}(u-\mu_Y)^2 f(u)du = \sigma^2/b^2$. Thus, the transformation $\widetilde{f}(Y)$ changes the variance of the random capacity in proportion to $\frac{1}{b^2}$. It follows that the expected profit $\widetilde{\mathbb{E}}(P)$ has an upper-bound $\overline{P}(\mu_Y)$ since $\lim_{a\to+\infty} \widetilde{f}(\mu_Y) = 1$. □

Proof of Corollaries 2.5.6, 2.5.7 and 2.5.8. To begin with, we analyze how

$S(Q^*)$ changes as the mean-capacity improvement $\Delta Y$ increases using Equation A.11.

$$
\begin{aligned}
S(Q^*) &= (1+r_f)^{-1}\left\{P_c(Q^*)[1-F(Q^*)] + \int_0^{Q^*} P(Y)f(Y)dY\right\} \\
&= (1+r_f)^{-1}\int_0^{+\infty} \overline{P}(Y)f(Y)dY \\
\widetilde{S}(Q^*) &= (1+r_f)^{-1}\int_0^{+\infty} \overline{P}(Y)f(Y-\Delta Y)dY \\
&= (1+r_f)^{-1}\int_0^{+\infty} \overline{P}(Y+\Delta Y)f(Y)dY \\
\frac{\mathrm{d}\widetilde{S}(Q^*)}{\mathrm{d}\Delta Y} &= (1+r_f)^{-1}\int_0^{+\infty} \overline{P}'(Y+\Delta Y)f(Y)dY \\
&= (1+r_f)^{-1}\int_0^{Q^*-\Delta Y} \overline{P}'(Y+\Delta Y)f(Y)dY \\
&= \frac{r-s+d}{1+r_f}\int_0^{Q^*-\Delta Y}\left[c_F - \Phi\left(\frac{Y-\mu_Z}{\sigma_Z}\right) - \delta_{MZ}s_R\phi\left(\frac{Y-\mu_Z}{\sigma_Z}\right)\right] f(Y)dY
\end{aligned}
$$

Note that increasing $\frac{\mathrm{d}\widetilde{S}(Q^*)}{\mathrm{d}\Delta Y}$ is a sufficient condition for increasing the benefit of mean-capacity improvement, namely $\Delta S(Q^*) = \int_0^{\Delta Y}\frac{\mathrm{d}\widetilde{S}(Q^*)}{\mathrm{d}\Delta Y}d\Delta Y$. Now we discuss how various parameter changes impact $\frac{\mathrm{d}\widetilde{S}(Q^*)}{\mathrm{d}\Delta Y}$. □

PROOF OF COROLLARY 2.5.6. We prove each result separately.
(a) Increasing $\delta_{MZ}$ reduces both $\left[c_F - \Phi\left(\frac{Y-\mu_Z}{\sigma_Z}\right) - \delta_{MZ}s_R\phi\left(\frac{Y-\mu_Z}{\sigma_Z}\right)\right]$ and $Q^*$ (Corollary 2.5.4), and hence reduces $\frac{\mathrm{d}\widetilde{S}(Q^*)}{\mathrm{d}\Delta Y}$.

When $\delta_{MZ} > 0$, increasing $s_R$ reduces both $\left[c_F - \Phi\left(\frac{Y-\mu_Z}{\sigma_Z}\right) - \delta_{MZ}s_R\phi\left(\frac{Y-\mu_Z}{\sigma_Z}\right)\right]$ and $Q^*$ (Corollary 2.5.4), and hence reduces $\frac{\mathrm{d}\widetilde{S}(Q^*)}{\mathrm{d}\Delta Y}$. When $\delta_{MZ} < 0$, increasing $s_R$ increases both $\left[c_F - \Phi\left(\frac{Y-\mu_Z}{\sigma_Z}\right) - \delta_{MZ}s_R\phi\left(\frac{Y-\mu_Z}{\sigma_Z}\right)\right]$ and $Q^*$ (Corollary 2.5.4), and hence increases $\frac{\mathrm{d}\widetilde{S}(Q^*)}{\mathrm{d}\Delta Y}$.

(b) (i) We have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\delta_{MZ}}\left(\frac{\mathrm{d}S(Q^*)}{\mathrm{d}b}\right) &= \frac{\mathrm{d}}{\mathrm{d}\delta_{MZ}}\int_{-\infty}^{bQ^*-(b-1)\mu_Y} P'\left(\frac{u-\mu_Y}{b}+\mu_Y\right)\left(-\frac{u-\mu_Y}{b^2}\right)f(u)du \\
&= \int_{-\infty}^{bQ^*-(b-1)\mu_Y} \frac{\mathrm{d}}{\mathrm{d}\delta_{MZ}}(r-s+d)\left[c_F-\Phi\left(\frac{u-(b-1)\mu_Y-\mu_Z}{b\sigma_Z}\right)\right. \\
&\quad \left. -\delta_{MZ}s_R\phi\left(\frac{u-(b-1)\mu_Y-\mu_Z}{b\sigma_Z}\right)\right]\cdot\left(-\frac{u-\mu_Y}{b^2}\right)f(u)du \\
&= (r-s+d)\int_{-\infty}^{bQ^*-(b-1)\mu_Y} s_R\phi\left[\frac{u-(b-1)\mu_Y-\mu_Z}{b\sigma_Z}\right]\cdot \\
&\quad \left(\frac{u-\mu_Y}{b^2}\right)f(u)du
\end{aligned}$$

When $Q^* \leq \mu_Y$, it follows that $bQ^*-(b-1)\mu_Y \leq \mu_Y$ and we arrive at $\frac{\mathrm{d}}{\mathrm{d}\delta_{MZ}}\left(\frac{\mathrm{d}S(Q^*)}{\mathrm{d}b}\right) < 0$.

(b) (ii) We have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s_R}\left(\frac{\mathrm{d}S(Q^*)}{\mathrm{d}b}\right) &= \frac{\mathrm{d}}{\mathrm{d}s_R}\int_{-\infty}^{bQ^*-(b-1)\mu_Y} P'\left(\frac{u-\mu_Y}{b}+\mu_Y\right)\left(-\frac{u-\mu_Y}{b^2}\right)f(u)du \\
&= \int_{-\infty}^{bQ^*-(b-1)\mu_Y} \frac{\mathrm{d}}{\mathrm{d}s_R}(r-s+d)\left[c_F-\Phi\left(\frac{u-(b-1)\mu_Y-\mu_Z}{b\sigma_Z}\right)\right. \\
&\quad \left. -\delta_{MZ}s_R\phi\left(\frac{u-(b-1)\mu_Y-\mu_Z}{b\sigma_Z}\right)\right]\cdot\left(-\frac{u-\mu_Y}{b^2}\right)f(u)du \\
&= (r-s+d)\int_{-\infty}^{bQ^*-(b-1)\mu_Y} \delta_{MZ}\phi\left[\frac{u-(b-1)\mu_Y-\mu_Z}{b\sigma_Z}\right]\cdot \\
&\quad \left(\frac{u-\mu_Y}{b^2}\right)f(u)du
\end{aligned}$$

When $Q^* \leq \mu_Y$, it follows that $bQ^*-(b-1)\mu_Y \leq \mu_Y$. We arrive at that $\frac{\mathrm{d}}{\mathrm{d}s_R}\left(\frac{\mathrm{d}S(Q^*)}{\mathrm{d}b}\right) < 0$ if $\delta_{MZ} > 0$ and $\frac{\mathrm{d}}{\mathrm{d}s_R}\left(\frac{\mathrm{d}S(Q^*)}{\mathrm{d}b}\right) > 0$ if $\delta_{MZ} < 0$. □

Proof of Corollary 2.5.7. It follows directly from Proposition 2.5.5(a). □

Proof of Corollary 2.5.8. (a) Increasing $c_F$ increases both

$$\left[c_F - \Phi\left(\frac{Y-\mu_Z}{\sigma_Z}\right) - \delta_{MZ} s_R \phi\left(\frac{Y-\mu_Z}{\sigma_Z}\right)\right]$$

and $Q^*$ (Corollary 2.5.4), and hence increases $\frac{\mathrm{d}\widetilde{S}(Q^*)}{\mathrm{d}\Delta Y}$.

(b) We have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}c_F}\left(\frac{\mathrm{d}S(Q^*)}{\mathrm{d}b}\right) &= \frac{\mathrm{d}}{\mathrm{d}c_F}\int_{-\infty}^{bQ^*-(b-1)\mu_Y} P'\left(\frac{u-\mu_Y}{b}+\mu_Y\right)\left(-\frac{u-\mu_Y}{b^2}\right) f(u)du \\
&= \int_{-\infty}^{bQ^*-(b-1)\mu_Y} \frac{\mathrm{d}}{\mathrm{d}c_F}(r-s+d)\Big[c_F - \Phi\left(\frac{u-(b-1)\mu_Y-\mu_Z}{b\sigma_Z}\right) \\
&\quad - \delta_{MZ} s_R \phi\left(\frac{u-(b-1)\mu_Y-\mu_Z}{b\sigma_Z}\right)\Big]\left(-\frac{u-\mu_Y}{b^2}\right) f(u)du \\
&= (r-s+d)\int_{-\infty}^{bQ^*-(b-1)\mu_Y}\left(-\frac{u-\mu_Y}{b^2}\right) f(u)du
\end{aligned}$$

If $bQ^* - (b-1)\mu_Y \leq \mu_Y \Leftrightarrow Q^* \leq \mu_Y$, we have that $\left(-\frac{u-\mu_Y}{b^2}\right) < 0$ for $u \in (-\infty, bQ^* - (b-1)\mu_Y)$, and it follows that $\frac{\mathrm{d}}{\mathrm{d}c_F}\left(\frac{\mathrm{d}S(Q^*)}{\mathrm{d}b}\right) > 0$. Otherwise, if $bQ^* - (b-1)\mu_Y > \mu_Y \Leftrightarrow Q^* > \mu_Y$, we have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}c_F}\left(\frac{\mathrm{d}S(Q^*)}{\mathrm{d}b}\right) &= (r-s+d)\int_{-\infty}^{bQ^*-(b-1)\mu_Y}\left(-\frac{u-\mu_Y}{b^2}\right) f(u)du \\
&> (r-s+d)\int_{-\infty}^{+\infty}\left(-\frac{u-\mu_Y}{b^2}\right) f(u)du = 0
\end{aligned}$$

□

APPENDIX B

PROOFS OF SECTION 3

Proof of Proposition 3.3.1. Let $OH$ and $BO$ respectively denote the number of units on hand and the number of units back-ordered. Note that at steady-state, the probability distribution of $OO$ is given by $p_{OO}(x) = (1-\rho)\rho^x$. From basic principles we have that since $OH(S) = S + 1 - OO + BO(S)$, $OH(S) \cdot BO(S) = 0$, $OH(S) \geq 0$, $BO(S) \geq 0$ and $I(S) = h \cdot OH(S) + b \cdot BO(S)$, we have the distribution of $p_{I(S)}(\cdot)$ in Equation 3.4. □

Proof of Proposition 3.3.2. When $\eta \leq h(S+1)$, we obtain

$$\begin{aligned}
\Omega(S,\eta) &= \sum_{k=0}^{k_0}[h(S+1-k)-\eta](1-\rho)\rho^k + \sum_{k=k_1}^{\infty}[b(k-S-1)-\eta](1-\rho)\rho^k, \\
&= (1-\rho^{k_0+1})[h(S+1)-\eta] + hk_0\rho^{k_0+1} - \frac{h}{1-\rho}(\rho-\rho^{k_0+1}) \\
&+ \rho^{k_1}[-\eta - b(S+1)] + bk_1\rho^{k_1} + \frac{b}{1-\rho}\rho^{k_1+1}, \\
&= [h(S+1)-\eta] - \rho^{k_0+1}[h(S+1)-\eta-hk_0-\frac{h}{1-\rho}] - \frac{h\rho}{1-\rho}, \\
&+ \rho^{k_1}[-\eta-b(S+1)+bk_1+\frac{b\rho}{1-\rho}], \\
&= [h(S+1)-\eta] - \rho^{k_0+1}\left[h\lceil\eta/h\rceil - \eta - \frac{h}{1-\rho}\right] - \frac{h\rho}{1-\rho} \\
&\quad + \rho^{k_1}\left[b\lceil\eta/b\rceil - \eta + \frac{b\rho}{1-\rho}\right].
\end{aligned}$$

Define $\Delta\Omega(S,\eta) = \Omega(S,\eta) - \Omega(S-1,\eta)$. Hence

$$\Delta\Omega(S,\eta) = h + (1-\rho)\rho^{k_0}\left\{\left[h\lceil\eta/h\rceil - \eta - \frac{h}{1-\rho}\right] - \rho^{k_1-k_0-1}\left[b\lceil\eta/b\rceil - \eta + \frac{b\rho}{1-\rho}\right]\right\}.$$

The value of $S$ which minimizes $\Omega(S,\eta)$ is then given by $\sup\{S \in N : \Delta\Omega(S,\eta) \leq 0\} = \max\{S_1^*(\eta), S_2^*(\eta)\}$, where $S_1^*(\eta)$ and $S_2^*(\eta)$ respectively are as given in Equations 3.12 and 3.13. Note that $\Omega(\hat{S}^*,\eta) = (1-\rho)(h\lceil\eta/h\rceil - \eta) + \rho^{\lceil\eta/h\rceil+\lceil\eta/b\rceil}[b\lceil\eta/b\rceil - \eta + \frac{b\rho}{1-\rho}]$ when $\hat{S}^* = \lceil\eta/h\rceil - 1$. Also note that $k_1 - k_0 = \lceil\eta/h\rceil + \lceil\eta/b\rceil$.

When $\eta > h(S+1)$, we obtain

$$\begin{aligned}\Omega(S,\eta) &= \sum_{k=k_1}^{\infty}[b(k-S-1)-\eta](1-\rho)\rho^k,\\ &= \rho^{k_1}[-\eta - b(S+1)] + bk_1\rho^{k_1} + \frac{b}{1-\rho}\rho^{k_1+1},\\ &= \rho^{k_1}[-\eta - b(S+1) + bk_1 + \frac{b\rho}{1-\rho}],\\ &= \rho^{k_1}\left[b\lceil\eta/b\rceil - \eta + \frac{b\rho}{1-\rho}\right],\end{aligned}$$

and it follows that $\Delta\Omega(S,\eta) = (1-\rho)\rho^{k_1-1}\left[b\lceil\eta/b\rceil - \eta + \frac{b\rho}{1-\rho}\right]$.

Since $b\lceil\eta/b\rceil - \eta + \frac{b\rho}{1-\rho} > 0$, we want to increase $k_1$ as much as possible. This yields the minimizer $S_3^*(\eta) = \lceil\eta/h\rceil - 2$. □

PROOF OF THEOREM 3.3.3. We begin the proof by presenting two results.

(1) Let $k_0 = k_0(S)$ be a function of $S$. We show that $k_0(S_2^*) \geq -1$ .

Let $\Phi = -\left[h\lceil\eta/h\rceil - \eta - \frac{h}{1-\rho}\right] + \rho^{k_1-k_0-1}\left[b\lceil\eta/b\rceil - \eta + \frac{b\rho}{1-\rho}\right]$. Note that $h\lceil\eta/h\rceil - \eta = h(\lceil\eta/h\rceil - \eta/h) < h$ and that $b\lceil\eta/b\rceil - \eta = b(\lceil\eta/b\rceil - \eta/b) \geq 0$. We have $\Phi \geq \frac{h}{1-\rho} - (h\lceil\eta/h\rceil - \eta) > \frac{h}{1-\rho} - h = \frac{h\rho}{1-\rho}$. Note that $\rho < 1$. It follows that $k_0(S_2^*) = \left\lfloor\frac{\ln(\frac{h}{1-\rho}) - \ln(\Phi)}{\ln(\rho)}\right\rfloor \geq \left\lfloor\frac{\ln(\frac{h}{1-\rho}) - \ln(\frac{h\rho}{1-\rho})}{\ln(\rho)}\right\rfloor = \left\lfloor\frac{-\ln(\rho)}{\ln(\rho)}\right\rfloor = -1$.

(2) We show that $S_2^* < S_1^* \Leftrightarrow \Omega(S_1^*, \eta) > \Omega(S_3^*, \eta)$.

$$S_2^* < S_1^* \Leftrightarrow \left\lfloor \frac{\ln(\frac{h}{1-\rho}) - \ln\left\{-\left[h\lceil\eta/h\rceil - \eta - \frac{h}{1-\rho}\right] + \rho^{k_1-k_0-1}\left[b\lceil\eta/b\rceil - \eta + \frac{b\rho}{1-\rho}\right]\right\}}{\ln(\rho)} \right\rfloor$$

$$< 0$$

$$\Leftrightarrow \ln\left(\frac{h}{1-\rho}\right) - \ln\left\{ - \left[h\lceil\eta/h\rceil - \eta - \frac{h}{1-\rho}\right] + \rho^{k_1-k_0-1}\left[b\lceil\eta/b\rceil - \eta + \frac{b\rho}{1-\rho}\right]\right\} > 0$$

$$\Leftrightarrow \ln\left(\frac{h}{1-\rho}\right) > \ln\left\{ - \left[h\lceil\eta/h\rceil - \eta - \frac{h}{1-\rho}\right] + \rho^{k_1-k_0-1}\left[b\lceil\eta/b\rceil - \eta + \frac{b\rho}{1-\rho}\right]\right\}$$

$$\Leftrightarrow \frac{h}{1-\rho} > -\left[h\lceil\eta/h\rceil - \eta - \frac{h}{1-\rho}\right] + \rho^{k_1-k_0-1}\left[b\lceil\eta/b\rceil - \eta + \frac{b\rho}{1-\rho}\right]$$

$$\Leftrightarrow 0 > -\left[h\lceil\eta/h\rceil - \eta\right] + \rho^{k_1-k_0-1}\left[b\lceil\eta/b\rceil - \eta + \frac{b\rho}{1-\rho}\right]$$

$$\Leftrightarrow \Omega(S_1^*, \eta) - \Omega(S_3^*, \eta) > 0.$$

It follows from result (2) that $S_2^*$ is preferred to both $S_1^*$ and $S_3^*$ when $S_2^* \geq S_1^*$. And when $S_2^* < S_1^*$, we have that $k_0(S_2^*) = S_2^* + 1 - \lceil\eta/h\rceil = S_2^* - S_1^* < 0$; based on result (1), we have that $k_0(S_2^*) = -1$ and thus $S_2^* = -1 + \lceil\eta/h\rceil - 1 = S_3^*$. Based on result (2), we have that when $S_2^* < S_1^*$, $\Omega(S_1^*, \eta) > \Omega(S_3^*, \eta)$ and $F_\beta(S_1^*, \eta) > F_\beta(S_3^*, \eta)$; $S_2^* = S_3^*$ is hence optimal. Therefore, $\hat{S}^* = S_2^*$ is the optimal restocking level for any given $\eta$. □

Proof of Theorem 3.3.4. We characterize $F_\beta(\eta)$ and consider four possible scenarios. Denote $x_+ = x + \Delta x$ where $\Delta x \to 0+$.

1) When $\eta$ increases from $vh$ to $vh_+$, suppose that $\lceil\eta/b\rceil$ remains unchanged, it can be shown that $k_0^*$ will decrease by 1, and that $\hat{S}^*$ and $k_1^*$ remain the same. Let $g(\eta) =$

$-\left[h\lceil\eta/h\rceil-\eta-\frac{h}{1-\rho}\right]+\rho^{\lceil\eta/h\rceil+\lceil\eta/b\rceil-1}\left[b\lceil\eta/b\rceil-\eta+\frac{b\rho}{1-\rho}\right]$. When $\eta$ increases from $vh$ to $vh_+$, we have that $g(vh)=\frac{h}{1-\rho}+\rho^{v+\lceil vh/b\rceil-1}\left[b\lceil vh/b\rceil-\eta+\frac{b\rho}{1-\rho}\right]=\rho^{-1}g(vh_+)$; it follows that $k_0^*$ will decrease by 1 since $k_0^*=\left\lfloor\{\ln(\frac{h}{1-\rho})-\ln[g(\eta)]\}/\ln(\rho)\right\rfloor$ based on Theorem 3.3.3. We now examine whether $F_\beta(\hat{S}^*,\eta)$ is continuous when $\eta$ increases from $vh$ to $vh_+$. We have

$$(1-\beta)\Delta F_\beta=-\rho^{k_0^*}\left[h-\frac{h}{1-\rho}\right]+\rho^{k_0^*+1}\cdot\frac{h}{1-\rho}=0$$

This means a smooth transition from $\eta=vh$ to $\eta=vh_+$ in the value of $F_\beta(\eta)$. Note that $\frac{\partial}{\partial\eta}F_\beta(\hat{S}^*,\eta)$ increases in this case and $\eta$ may be a local minimum.

2) When $\eta$ increases from $\eta=wb$ to $\eta=wb_+$, suppose that $\lceil\eta/h\rceil=v$ remains unchanged, it can be shown that $k_0^*$ remains unchanged since $g(wb)=g(wb_+)$ and $k_1^*$ increase by 1. We have that

$$\begin{aligned}(1-\beta)\Delta F_\beta&=\rho^{v+w+1-1}\left(b+\frac{b\rho}{1-\rho}\right)-\rho^{v+w-1}\left(\frac{b\rho}{1-\rho}\right)\\&=\rho^{v+w-1}\left(\frac{b\rho}{1-\rho}-\frac{b\rho}{1-\rho}\right)=0\end{aligned}$$

This indicates a smooth transition from $\eta=wb$ to $\eta=wb_+$ in the value of $F_\beta(\eta)$. Note that $\frac{\partial}{\partial\eta}F_\beta(\hat{S}^*,\eta)$ increases in this case and $\eta$ may be a local minimum.

Note that if $\eta$ increases from $vh$ to $vh_+$ and from $\eta=wb$ to $\eta=wb_+$ at the same time, $\Delta F_\beta$ can be decomposed into a cowbination of 1) and 2) and shown to be zero as well. Note that $\frac{\partial}{\partial\eta}F_\beta(\hat{S}^*,\eta)$ increases in this case and $\eta$ may be a local minimum.

3) Suppose $\eta$ increases from $\eta_0$ to $\eta_{0+}$ (let $\lceil\eta_0/h\rceil=v+1$) and causes $k_0^*$ to increase by 1 ($\lceil\eta/h\rceil$ and $\lceil\eta/b\rceil$ remain the same, $\hat{S}^*$ increases from $S_0$ to $S_0+1$) and $k_1^*$ to increase by 1. It must be that $k_0^*(\eta_{0+})=\left\lfloor\{\ln(\frac{h}{1-\rho})-\ln[g(\eta_{0+})]\}/\ln(\rho)\right\rfloor=\{\ln(\frac{h}{1-\rho})-$

$\ln[g(\eta_{0+})]\}/\ln(\rho)$. It follows that $\Delta\Omega(S_0+1,\eta_{0+}) = h + (1-\rho)\rho^{k_0^*(\eta_{0+})}g(\eta_{0+}) = 0$ and $F_\beta(S_0,\eta_{0+}) = F_\beta(S_0+1,\eta_{0+})$. It follows that $F_\beta(S_0,\eta_0) = F_\beta(S_0+1,\eta_0) = F_\beta(S_0+1,\eta_{0+})$. This means a smooth transition from $\eta=\eta_0$ to $\eta=\eta_{0+}$ in the value of $F_\beta(\eta)$. Note that $\frac{\partial}{\partial\eta}F_\beta(\hat{S}^*,\eta)$ increases in this case and hence $\eta_0$ is not a local minimum. It is worth noting that since the transition from $\eta=vh$ to $\eta=vh_+$ falls in case 1), we have that $k_0^*$ remains constant during the jump from $\eta=vh$ to $\eta=(v+1)h$ for any $n$ that satisfies case 3).

4) For any continuous interval that both $k_0^*$ and $k_1^*$ remain the same, we have Equation 3.14. Thus, the function $F_\beta(\hat{S}^*,\eta)$ is linear in this interval when $k_0^*$ and $k_1^*$ remain the same. Thus, no local minimum can be found in this interval unless $\partial F_\beta/\partial\eta$ is exactly zero (we do not consider this rare special case).

In sum, we have shown that $F_\beta(\hat{S}^*,\eta)$ is a continuous function with respect to $\eta$. Moreover, the local minimums of $F_\beta(\hat{S}^*,\eta)$ are located at $\eta=vh$ and $\eta=wb$. □

Proof of Lemma 3.3.5. It is straightforward to show that when $n\le q-1$, $y(\eta)$ is decreasing in $\eta=(mq+n)h$. We continue to show that the same result applies when $n=q$.

$$\begin{aligned}\Delta y(\eta) &= \rho^{(mq+q+1)+(m+2)}[q+(\rho-1)+\rho] - \rho^{(mq+q)+(m+1)}[q+(\rho-1)q+\rho]\\ &= -\rho^{(mq+q)+(m+1)+1}[q+1-(\rho q+2\rho^2-\rho)]\\ &= -\rho^{(mq+q)+(m+1)+1}[(1-\rho)q+(1+\rho-2\rho^2)]\end{aligned}$$

Note that $\Delta y(\eta)<0$ when $\rho\in(0,1)$. Thus, $y(\eta)$ is always decreasing in $\eta=(mq+n)h$. □

Proof of Theorem 3.3.6. (1) We first show that $k_0^*\ge 0$ when $\eta=nh$. Since $g(nh)>\frac{h}{1-\rho}$ and $k_0^*(\eta)=\left\lfloor\frac{\ln(\frac{h}{1-\rho})-\ln\{g(\eta)\}}{\ln(\rho)}\right\rfloor$, we have $k_0^*(nh)\ge\left\lfloor\frac{\ln(\frac{h}{1-\rho})-\ln(\frac{h}{1-\rho})}{\ln(\rho)}\right\rfloor=0$.

(2) We then show that a necessary condition for $\eta = \hat{\eta}^*$ is that $k_0^* = 0$. Since $1-\rho \geq 1-\beta$, we have that $\frac{\partial}{\partial\eta}F_\beta(\hat{S}^*,\eta) > 0$ is possible only when $k_0^* = -1$. From the proof of Theorem 3.3.4, we have that $k_0^* = -1$ is only possible when $\eta$ increases from $nh$ to $nh_+$ and $k_0$ decreases by 1 from $k_0^* = 0$. Thus based on Theorem 3.3.4, it must be that $k_0^* = 0$ when $\eta = \hat{\eta}^*$.

Based on (1), (2) and Theorem 3.3.4, we only consider the case when $k_0 = 0$ and $\eta = (mq+n)h$. There are two cases:

Case 1: $\eta$ increases from $(mq+n)h$ to $(mq+n+1)h$ and $1 \leq n \leq q-1$. Based on the proof of Theorem 3.3.4, $k_0^*$ will not increase. Based on (1), we know that $k_0^*$ will remain zero.

$$\begin{aligned}\Delta\Omega(\hat{S}^*,\eta) &= \rho^{(mq+n+1)+m+1}[(q-n-1)h + \frac{b\rho}{1-\rho}] - \rho^{mq+n+m+1}\left[(q-n)h + \frac{b\rho}{1-\rho}\right] \\ &= \rho^{mq+n+m+1}\left[\rho(q-n-1) + \frac{q\rho^2}{1-\rho} - (q-n) - \frac{q\rho}{1-\rho}\right]h \\ &= -\rho^{mq+n+m+1}[q+(\rho-1)n+\rho]h\end{aligned}$$

Let $\Delta F_\beta = h + (1-\beta)^{-1}\Delta EL \geq 0$, we obtain $\rho^{mq+n+m+1}[q+(\rho-1)n+\rho] \leq 1-\beta$.

Case 2: $\eta$ increases from $(mq)h$ to $(mq+1)h$. Note $k_0^*$ will remain zero based on (1).

$$\begin{aligned}\Delta\Omega(\hat{S}^*,\eta) &= \rho^{(mq+1)+m+1}\left[(q-1)h + \frac{b\rho}{1-\rho}\right] - \rho^{mq+m}\left[0 + \frac{b\rho}{1-\rho}\right] \\ &= \rho^{mq+m}\left[-\frac{q\rho^2}{1-\rho}(\rho^2-1) + (q-1)\rho^2\right]h \\ &= \rho^{mq+m}[-q\rho(1+\rho) + (q-1)\rho^2]h \\ &= -\rho^{mq+m+1}(q+\rho)h\end{aligned}$$

which is the same expression in Case 1 when $n = 0$. We can therefore obtain the condition $\rho^{mq+n+m+1}[q+(\rho-1)n+\rho] \le 1-\beta$ in general for $0 \le n \le q-1$. It follows from Lemma 3.3.5 that there is one and only one local minimum of $F_\beta(\hat{S}^*, \eta)$ that satisfies $\hat{\eta}^* = \inf\{\eta = (mq+n)h \;:\; \rho^{mq+n+m+1}[q+(\rho-1)n+\rho] \le 1-\beta\}$, which must be the global minimum. We also have that $\hat{S}^* + 1 = \lceil \hat{\eta}^*/h \rceil$ based on (2) and arrive at Equation 3.15. □

Proof of Corollary 3.3.7. The inequality below gives the upper bound and lower bound of $y(\eta)$ for a given $\eta$.

$$\rho^{mq+n+m+1}(\rho q+1) \le y(\eta) \le \rho^{mq+n+m+1}(q+\rho) \tag{B.1}$$

Based on Theorem 3.3.6, we have one candidate of the optimal solution, whose upper bound of $y(\eta)$ is not higher than $1-\beta$, that is, $\rho^{m'q+n'+m'+1}(q+\rho) \le 1-\beta$. We have that

$$m'(q+1) + n' + 1 = \left\lceil \frac{\ln(q+\rho) - \ln(1-\beta)}{-\ln\rho} \right\rceil$$

$$\textit{where}$$

$$m' = \left\lfloor \left\lceil \frac{\ln(q+\rho) - \ln(1-\beta)}{-\ln\rho} \right\rceil / (q+1) \right\rfloor$$

$$n' = \max\left\{0, \left\lceil \frac{\ln(q+\rho) - \ln(1-\beta)}{-\ln\rho} \right\rceil - m'(q+1) - 1\right\}$$

And $\eta' = (m'q+n')h$ and $S'+1 = m'q+n' = \left\lceil \frac{\ln(q+\rho)-\ln(1-\beta)}{-\ln\rho} \right\rceil - m' - 1$. Another candidate of the optimal solution is $\eta'' = (m'q+n'-1)h$ and $S''+1 = m'q+n'-1$. If $y(\eta'') \le 1-\beta$, then $(S'', \eta'')$ is the optimal solution; otherwise, $(S', \eta')$ is the optimal solution. Thus, $S''$ and $S'$ are the lower-bound and upper-bound of $\hat{S}^*$, respectively. □

Proof of Proposition 3.4.1. Let $\beta_1 > \beta_2$. We have that $1-\beta_1 < 1-\beta_2$. Denote $S_1 = \hat{S}^*|_{\beta=\beta_1}$ and $S_2 = \hat{S}^*|_{\beta=\beta_2}$ as given in Theorem 3.3.6. Denote $\eta_1 = \inf\{\eta = (mq+n) : y(\eta) \le 1-\beta_1\}$ and $\eta_2 = \inf\{\eta = (mq+n) : y(\eta) \le 1-\beta_2\}$ and correspondingly $S_1 = (\eta_1 - 1)h$ and $S_2 = (\eta_2 - 1)h$. We have that $\eta_1 \ge \eta_2$ since $y(\eta)$ decreases in $\eta$ (Lemma 3.3.5). Hence, $S_1 \ge S_2$ and the optimal base-stock level is non-decreasing with increase in $\beta$. □

Proof of Proposition 3.4.2. We first observe that $y(\eta) = \rho^{mq+n+m+1}[q+(\rho-1)n+\rho] = \rho^{\hat{S}^*+m+2}[q+(\rho-1)n+\rho]$ is monotonically increasing in $q$ for any given $\hat{S}^*$ (or equivalently, any given $\eta$). Let $q_1 > q_2$. Denote $S_1 = \hat{S}^*|_{q=q_1}$ and $S_2 = \hat{S}^*|_{q=q_2}$ as given in Theorem 3.3.6. Denote $\eta_1 = (S_1+1)h$ and $\eta_2 = (S_2+1)h$. We have $y(\eta_2)|_{q=q_2} \le 1-\beta$ and $y(\eta_2+h)|_{q=q_2} > 1-\beta$. We also have $y(\eta_2+h)|_{q=q_1} > y(\eta_2+h)|_{q=q_2} > 1-\beta$ since $y(\eta)$ is monotonically increasing in $q$. Thus, $\eta_1 < \eta_2 + h$ and it follows that $\eta_1 \le \eta_2$, $S_1 \le S_2$ and $\hat{B}^*$ is non-decreasing in $q$. □

Proof of Proposition 3.4.3. We first establish that $y(\eta) = \rho^{mq+n+m+1}[q+(\rho-1)n+\rho]$ is monotonically increasing in $\rho$ for any given $\hat{S}^*$ (or equivalently, any given $\eta$) since $\partial y(\eta)/\partial\rho = \rho^{mq+n+m}(mq+n+m+1)[q+(\rho-1)n+\rho] + \rho^{mq+n+m+1}(n+1) > 0$. Let $\rho_1 > \rho_2$. Denote $S_1 = \hat{S}^*|_{\rho=\rho_1}$ and $S_2 = \hat{S}^*|_{\rho=\rho_2}$ as given in Theorem 3.3.6. Denote $\eta_1 = (S_1+1)h$ and $\eta_2 = (S_2+1)h$. We have $y(\eta_2)|_{\rho=\rho_2} \le 1-\beta$ and $y(\eta_2+h)|_{\rho=\rho_1} > y(\eta_2+h)|_{\rho=\rho_2} > 1-\beta$. It follows that $\eta_1 < \eta_2 + h$ and it follows that $\eta_1 \le \eta_2$, $S_1 \le S_2$ and $\hat{B}^*$ is non-decreasing in $\rho$. □

Proof of Proposition 3.3.8. Based on Theorem 3.3.6, when $m = 0$ and $n = 0$, we have $y(\eta) = \rho(q+\rho) < 1-\beta$. Since $\hat{S}^* = \inf\{\eta = (mq+n)-1 : y(\eta) \le 1-\beta\}$, whenever $\rho(q+\rho) < 1-\beta$ is satisfied, we have $\hat{S}^* = -1$ and the optimal base-stock level is zero. □

Proof of Proposition 3.5.1. When $\beta = 0$, we have that $\hat{\eta}_0(S) = \min[\hat{K}(S)]$ and CVaR of $\hat{K}(S)$ converges to the expected value of $\hat{K}(S)$ when $\beta = 0$, as shown in Equation B.2.

$$\hat{\zeta}_0(S) = \int_{\hat{K}(S) \geq \hat{\eta}_0(S)} \hat{K}(S) p_{\hat{K}(S)}(y) dy = \mathbb{E}[\hat{K}(S)] = \mathbb{E}[r(S)] \tag{B.2}$$

It follows from Equation B.2 that $\hat{S}^*|_{\beta=0} = \arg\min_S \hat{\zeta}_0(S) = \arg\min_S \mathbb{E}[r(S)] = S^*_{EC}$. Thus, $\hat{B}^*$ converges to $B^*_{EC}$ when $\beta = 0$. Moreover, according to Proposition 3.4.1, we have that $\hat{B}^* \geq B^*_{EC}$. □

# APPENDIX C

## PROOFS OF SECTION 4

Proof of Lemma 4.3.1. When $\rho = 1$, we have

$$\Delta\widetilde{P}_0^A(r) = \widetilde{P}_0^A(r+1) - \widetilde{P}_0^A(r) = -\widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)\cdot\frac{ca_R}{ca_R+b_R-b}D$$

$$\begin{aligned}\Delta\widetilde{P}_{c+r}^A = \widetilde{P}_{c+r+1}^A - \widetilde{P}_{c+r}^A &= \Delta\widetilde{P}_0^A(r)\cdot\frac{1}{ca}[b-c(a-a_R)]D \\ &= -\widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)\cdot\frac{ca_R}{ca_R+b_R-b}D\cdot\frac{1}{ca}[b-c(a-a_R)]D \\ &= -\widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)D^2\cdot\frac{a_R}{a}\cdot\frac{b-c(a-a_R)}{ca_R+b_R-b}\end{aligned}$$

$$\begin{aligned}\Delta\widetilde{L}_q &= \Delta\widetilde{P}_c^A\left(\frac{1}{2}r^2+\frac{1}{2}r-\lambda a_R r\right)+\widetilde{P}_c^A(r+1)\left(r+1-\lambda a_R\right)+\widetilde{P}_{c+r+1}^A+r\cdot\Delta\widetilde{P}_{c+r}^A \\ &= -\widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)\cdot\frac{ca_R}{ca_R+b_R-b}D\cdot\frac{1}{ca_R+b_R-b}\cdot[b-c(a-a_R)]D \\ &\cdot\left(\frac{1}{2}r^2+\frac{1}{2}r-\lambda a_R r\right) \\ &+\widetilde{P}_0^A(r+1)\cdot\frac{1}{ca_R+b_R-b}\cdot[b-c(a-a_R)]D\left(r+1-\lambda a_R\right) \\ &+\widetilde{P}_0^A(r+1)\cdot\frac{1}{ca}\cdot[b-c(a-a_R)]D-\widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)D^2\cdot\frac{a_R}{a}\cdot\frac{b-c(a-a_R)}{ca_R+b_R-b}\cdot r\end{aligned}$$

$$
\begin{aligned}
\Delta\widetilde{L}_q = \widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)D^2 \cdot \Bigg\{ &-\frac{ca_R\cdot[b-c(a-a_R)]}{(ca_R+b_R-b)^2}\cdot\left(\frac{1}{2}r^2+\frac{1}{2}r-\lambda a_R r\right) \\
&+\frac{b-c(a-a_R)}{ca_R+b_R-b}\cdot\left[B/D+\frac{ca_R}{ca_R+b_R-b}\frac{b_R-c(a-a_R)}{ca}+\frac{ca_R}{ca_R+b_R-b}r\right] \\
\cdot(r+1-\lambda a_R) \\
&+\frac{b-c(a-a_R)}{ca}\left[B/D+\frac{ca_R}{ca_R+b_R-b}\frac{b_R-c(a-a_R)}{ca}+\frac{ca_R}{ca_R+b_R-b}r\right] \\
&-\frac{a_R}{a}\cdot\frac{b-c(a-a_R)}{ca_R+b_R-b}\cdot r\Bigg\} \\
= \widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)D^2\cdot\frac{ca_R\cdot[b-c(a-a_R)]}{ca_R+b_R-b} \\
\cdot\Bigg\{ &-\frac{1}{ca_R+b_R-b}\left(\frac{1}{2}r^2+\frac{1}{2}r-\lambda a_R r\right) \\
&+\left[\frac{B}{ca_RD}+\frac{1}{ca_R+b_R-b}\frac{b_R-c(a-a_R)}{ca}+\frac{1}{ca_R+b_R-b}r\right](r+1-\lambda a_R) \\
&+\frac{1}{ca}\left[\frac{B(ca_R+b_R-b)}{ca_RD}+\frac{b_R-c(a-a_R)}{ca}+\frac{ca_R}{ca_R+b_R-b}r\right]-\frac{1}{ca}\cdot r\Bigg\} \\
= \widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)D^2\cdot\frac{ca_R\cdot[b-c(a-a_R)]}{ca_R+b_R-b}\cdot\Bigg\{ &\frac{1}{2(ca_R+b_R-b)}r^2 \\
&+\left[\frac{B}{ca_RD}+\frac{1}{ca_R+b_R-b}\left(\frac{b_R-c(a-a_R)}{ca}+\frac{1}{2}\right)\right]r \\
&+\left[\frac{B}{ca_RD}+\frac{b_R-c(a-a_R)}{ca(ca_R+b_R-b)}\right]\left(1-\lambda a_R+\frac{ca_R+b_R-b}{ca}\right)\Bigg\}
\end{aligned}
$$

We see that the sign of $\frac{\partial}{\partial r}\widetilde{L}_q$ depends on a quadratic function. Similarly, we continue to show that the sign of $\frac{\partial}{\partial r}\widetilde{\Pi}$ depends on a quadratic function and thus can be optimized analytically, where $\widetilde{\Pi} = C_1\widetilde{L}_q + C_2\lambda\widetilde{P}_{c+r}^A$.

$$
\begin{aligned}
\Delta\widetilde{\Pi} &= -C_1\Delta\widetilde{L}_q - C_2\lambda\Delta\widetilde{P}_{c+r}^A \\
&= -\widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)D^2\cdot\frac{ca_R\cdot[b-c(a-a_R)]}{ca_R+b_R-b}\cdot\Bigg\{\frac{C_1}{2(ca_R+b_R-b)}r^2 \\
&+\left[\frac{B}{ca_RD}+\frac{1}{ca_R+b_R-b}\left(\frac{b_R-c(a-a_R)}{ca}+\frac{1}{2}\right)\right]C_1\cdot r \\
&+C_1\left[\frac{B}{ca_RD}+\frac{b_R-c(a-a_R)}{ca(ca_R+b_R-b)}\right]\left(1-\lambda a_R+\frac{ca_R+b_R-b}{ca}\right)-C_2\lambda\cdot\frac{1}{ca}\Bigg\}
\end{aligned}
$$

Note that the above applies when $r \geq 0$. $\square$

Proof of Lemma 4.3.2. When $\rho \neq 1$, we have

$$\Delta\widetilde{P}_0^A(r) = \widetilde{P}_0^A(r+1) - \widetilde{P}_0^A(r) = \widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1) \cdot \frac{\rho}{1-\rho}(\lambda a_R - 1 + \rho)(\phi - 1)D\phi^r$$

$$\begin{aligned}
\Delta\widetilde{P}_{c+r}^A &= \widetilde{P}_{c+r+1}^A - \widetilde{P}_{c+r}^A \\
&= \left[\widetilde{P}_0^A(r+1)\phi^{r+1} - \widetilde{P}_0^A(r)\phi^r\right]\frac{b - c(a - a_R)}{ac}D \\
&= \left[\Delta\widetilde{P}_0^A(r)\phi^{r+1} + \widetilde{P}_0^A(r)(\phi - 1)\phi^r\right]\frac{b - c(a - a_R)}{ac}D \\
&= \widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1) \cdot \frac{\rho}{1-\rho}(\lambda a_R - 1 + \rho)D \cdot (\phi - 1)\phi^r \cdot \phi^{r+1}\frac{b - c(a - a_R)}{ac}D \\
&\quad + \widetilde{P}_0^A(r) \cdot (\phi - 1)\phi^r \cdot \frac{b - c(a - a_R)}{ac}D \\
&= \widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)D^2(\phi - 1)\phi^r\frac{b - c(a - a_R)}{ca} \\
&\quad \cdot \left[\frac{\rho}{1-\rho}(\lambda a_R - 1 + \rho)\phi^{r+1} + \frac{B}{D} + \frac{1 - \rho\phi^{r+1}}{1-\rho}(\lambda a_R - 1 + \rho)\right] \\
&= \widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)D^2(\phi - 1)\phi^r \cdot \frac{b - c(a - a_R)}{ca} \cdot \left[\frac{B}{D} + \frac{1}{1-\rho}(\lambda a_R - 1 + \rho)\right]
\end{aligned}$$

When $\phi < 1$, $\widetilde{P}_{c+r}^A$ decreases in $r$ with reasonable parameter values.

$$\begin{aligned}
\Delta\widetilde{L}_q(r) &= \Delta\widetilde{P}_c^A(r) \cdot \left[\frac{1 - (r+1)\phi^r + r\phi^{r+1}}{(1-\phi)^2} - \lambda a_R \cdot \frac{1 - \phi^r}{1 - \phi}\right] \\
&\quad + \widetilde{P}_c^A(r+1) \cdot \left[\frac{-(r+2)\phi^{r+1} + (r+1)\phi^{r+2} + (r+1)\phi^r - r\phi^{r+1}}{(1-\phi)^2}\right. \\
&\quad \left. - \lambda a_R \cdot \frac{(1 - \phi^{r+1}) - (1 - \phi^r)}{1 - \phi}\right] + \Delta\widetilde{P}_{c+r}^A \cdot r + \widetilde{P}_{c+r+1}^A
\end{aligned}$$

$$
\begin{aligned}
\Delta\widetilde{L}_q(r) = {} & \widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)\cdot\frac{\rho}{1-\rho}(\lambda a_R - 1 + \rho)D\cdot(\phi-1)\phi^r\cdot\frac{b-c(a-a_r)}{ca_R+b_R-b}D \\
& \cdot\left[\frac{1-(r+1)\phi^r + r\phi^{r+1}}{(1-\phi)^2} - \lambda a_R\cdot\frac{1-\phi^r}{1-\phi}\right] \\
& + \widetilde{P}_0^A(r+1)\cdot\frac{b-c(a-a_R)}{ca_R+b_R-b}D\phi^r\cdot[(r+1)-\lambda a_R] \\
& + \widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)D^2(\phi-1)\phi^r\cdot\frac{b-c(a-a_R)}{ca}\cdot\left[\frac{B}{D}+\frac{1}{1-\rho}(\lambda a_R - 1+\rho)\right]\cdot r \\
& + \widetilde{P}_0^A(r+1)\cdot\frac{b-c(a-a_R)}{ac}D\phi^{r+1} \\
= {} & \widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)D^2\phi^r[b-c(a-a_R)]\cdot\Bigg\{ \\
& \frac{\rho}{1-\rho}\frac{(\lambda a_R-1+\rho)(\phi-1)}{ca_R+b_R-b}\cdot\left[\frac{1-(r+1)\phi^r+r\phi^{r+1}}{(1-\phi)^2}-\lambda a_R\cdot\frac{1-\phi^r}{1-\phi}\right] \\
& + \frac{1}{ca_R+b_R-b}\cdot\left[\frac{B}{D}+\frac{1-\rho\phi^r}{1-\rho}(\lambda a_R-1+\rho)\right]\cdot[(r+1)-\lambda a_R] \\
& + \frac{\phi-1}{ca}\cdot\left[\frac{B}{D}+\frac{1}{1-\rho}(\lambda a_R-1+\rho)\right]\cdot r + \frac{\phi}{ca}\cdot\left[\frac{B}{D}+\frac{1-\rho\phi^r}{1-\rho}(\lambda a_R-1+\rho)\right]\Bigg\} \\
= {} & \widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)D^2\phi^r[b-c(a-a_R)]\cdot\Bigg\{ \\
& \frac{\rho}{1-\rho}\frac{(\lambda a_R-1+\rho)}{ca_R+b_R-b}\cdot\left[\frac{1-\phi^{r+1}}{\phi-1}+(r+1)\phi^r-\lambda a_R\phi^r+\lambda a_R\right] \\
& + \frac{1}{ca_R+b_R-b}\cdot\left[\frac{B}{D}+\frac{1}{1-\rho}(\lambda a_R-1+\rho)\right]\cdot[(r+1)-\lambda a_R] \\
& + \frac{1}{ca_R+b_R-b}\cdot\frac{-\rho\phi^r}{1-\rho}(\lambda a_R-1+\rho)\cdot[(r+1)-\lambda a_R] \\
& + \frac{\phi-1}{ca}\cdot\left[\frac{B}{D}+\frac{1}{1-\rho}(\lambda a_R-1+\rho)\right]\cdot r + \frac{\phi}{ca}\cdot\left[\frac{B}{D}+\frac{1-\rho\phi^r}{1-\rho}(\lambda a_R-1+\rho)\right]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\Delta \widetilde{L}_q(r) = \widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)D^2\phi^r[b-c(a-a_R)] \cdot \Bigg\{ \\
& \frac{\rho}{1-\rho}\frac{(\lambda a_R - 1 + \rho)}{ca_R + b_R - b} \cdot \left[\frac{\phi^{r+1}}{1-\phi} + \frac{1}{\phi - 1} + \lambda a_R\right] \\
& + \frac{1}{ca_R + b_R - b} \cdot \left[\frac{B}{D} + \frac{1}{1-\rho}(\lambda a_R - 1 + \rho)\right] \cdot [(r+1) - \lambda a_R] \\
& + \frac{\phi - 1}{ca} \cdot \left[\frac{B}{D} + \frac{1}{1-\rho}(\lambda a_R - 1 + \rho)\right] \cdot r + \frac{\phi}{ca}\frac{B}{D} + \frac{\phi}{ca}\frac{1-\rho\phi^r}{1-\rho}(\lambda a_R - 1 + \rho)\Bigg\} \\
& = \widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)D^2\phi^r[b-c(a-a_R)] \cdot \Bigg\{ \\
& \frac{\rho}{1-\rho}(\lambda a_R - 1 + \rho)\left[\frac{1}{1-\phi}\frac{1}{ca_R + b_R - b} - \frac{1}{ca}\right]\phi^{r+1} \\
& + \frac{\rho}{1-\rho}\frac{(\lambda a_R - 1 + \rho)}{ca_R + b_R - b} \cdot \left[-\frac{1}{1-\phi} + \lambda a_R\right] \\
& + \frac{1}{ca_R + b_R - b} \cdot \left[\frac{B}{D} + \frac{1}{1-\rho}(\lambda a_R - 1 + \rho)\right] \cdot [(r+1) - \lambda a_R] \\
& + \frac{\phi - 1}{ca} \cdot \left[\frac{B}{D} + \frac{1}{1-\rho}(\lambda a_R - 1 + \rho)\right] \cdot r + \frac{\phi}{ca}\left[\frac{B}{D} + \frac{1}{1-\rho}(\lambda a_R - 1 + \rho)\right]\Bigg\} \\
& = \widetilde{P}_0^A(r)\widetilde{P}_0^A(r+1)D^2\phi^r[b-c(a-a_R)] \cdot \Bigg\{ \\
& \frac{\rho}{1-\rho}(\lambda a_R - 1 + \rho)\left[\frac{1}{1-\phi}\frac{1}{ca_R + b_R - b} - \frac{1}{ca}\right]\phi^{r+1} \\
& \left[\frac{\phi - 1}{ca} + \frac{1}{ca_R + b_R - b}\right] \cdot \left[\frac{B}{D} + \frac{1}{1-\rho}(\lambda a_R - 1 + \rho)\right] \cdot r \\
& + \left[\frac{1 - \lambda a_R}{ca_R + b_R - b} + \frac{\phi}{ca}\right] \cdot \left[\frac{B}{D} + \frac{1}{1-\rho}(\lambda a_R - 1 + \rho)\right] \\
& + \frac{\rho}{1-\rho}\frac{(\lambda a_R - 1 + \rho)}{ca_R + b_R - b} \cdot \left[-\frac{1}{1-\phi} + \lambda a_R\right]\Bigg\}
\end{aligned}
$$

We see that the sign of $\frac{\partial}{\partial r}\widetilde{L}_q$ depends on a transcendental function. Similarly, we continue to show that the sign of $\frac{\partial}{\partial r}\widetilde{\Pi}$ depends on a transcendental function and thus can be optimized with Taylor approximations, where $\widetilde{\Pi} = C_1\widetilde{L}_q + C_2\lambda\widetilde{P}_{c+r}^A$.

$$
\begin{aligned}
\Delta\widetilde{\Pi} &= -C_1\Delta\widetilde{L}_q - C_2\lambda\Delta\widetilde{P}^A_{c+r} \\
&= -\widetilde{P}^A_0(r)\widetilde{P}^A_0(r+1)D^2\phi^r[b-c(a-a_R)]\cdot\Bigg\{ \\
&\frac{\rho}{1-\rho}(\lambda a_R - 1+\rho)\left[\frac{1}{1-\phi}\frac{1}{ca_R+b_R-b}-\frac{1}{ca}\right]C_1\phi^{r+1} \\
&\left[\frac{\phi-1}{ca}+\frac{1}{ca_R+b_R-b}\right]\cdot\left[\frac{B}{D}+\frac{1}{1-\rho}(\lambda a_R-1+\rho)\right]C_1\cdot r \\
&+\left[\left(\frac{1-\lambda a_R}{ca_R+b_R-b}+\frac{\phi}{ca}\right)C_1+\frac{\phi-1}{ca}\lambda C_2\right]\cdot\left[\frac{B}{D}+\frac{1}{1-\rho}(\lambda a_R-1+\rho)\right] \\
&+\frac{\rho}{1-\rho}\frac{(\lambda a_R-1+\rho)}{ca_R+b_R-b}\cdot\left[-\frac{1}{1-\phi}+\lambda a_R\right]C_1\Bigg\}
\end{aligned}
$$

□